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Influence of Coulomb scattering of electrons and holes between Landau levels on energy spectrum and collective properties of two-dimensional magnetoexcitons

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Abstract

This study is concerned with a two-dimensional electron–hole system in a strong perpendicular magnetic field with special attention devoted to the influence of the virtual quantum transitions of interacting particles between the Landau levels. It is shown that virtual quantum transitions of two Coulomb interacting particles from the lowest Landau levels to excited Landau levels with arbitrary quantum numbers n and m and their transition back to the lowest Landau levels in the second order of the perturbation theory result in indirect attraction between the particles supplementary to their Coulomb interaction. The influence of this indirect interaction on the chemical potential of the Bose–Einstein condensed magnetoexcitons and on the ground state energy of the metallic-type electron–hole liquid (EHL) is investigated in the Hartree–Fock approximation. The supplementary electron–electron and hole–hole interactions being averaged with direct pairing of operators increases the binding energy of magnetoexciton and the energy per pair in the EHL phase. The terms obtained in the exchange pairing of operators give rise to repulsion. Together with the Bogoliubov self-energy terms arising from the electron–hole supplementary interaction they both influence in the favor of BEC of magnetoexcitons with small momentum. The influence of the excited exciton bands on the energy spectrum and on the wave function of the lowest magnetoexciton band is studied in the second order of the perturbation theory. The BEC of magnetoexcitons in the superposition state is considered. The generalized Bogoliubov transformations, the BCS-type ground state wave function and the phase-space filling factors of the lowest and first excited Landau levels are determined.

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1. Introduction

Properties of atoms and excitons are dramatically changed in strong magnetic fields, such that the distance between Landau levels $\hbar\omega_c$, exceeds the corresponding Rydberg energies R_y and the magnetic length $l = \sqrt{\hbar c/eH}$ is small compared to their Bohr radii [1,2]. Even more interesting phenomena are exhibited in the case of two-dimensional (2D) electron systems due to the quenching of the kinetic energy at high magnetic fields, with the representative example being integer and fractional Quantum Hall effects [3–5]. The discovery of the FQHE [6–8] changed fundamentally the established concepts about charged elementary excitations in solids [5]. The notion of the incompressible quantum liquid (IQL) was introduced in Ref. [7] as a homogeneous phase with the quantized

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densities $\nu = p/q$, where p is an integer and $q \neq 1$ is odd having charged elementary excitations with a fractional charge $e^* = \pm e/q$. These quasiparticles were named as anyons. A classification for free anyons and their hierarchy were studied in [9,10]. An alternative concept to hierarchical scheme was proposed in [11], where the notion of composite fermions (CF) was introduced. The CF consists from the electron bound to an even number of flux quanta.In the frame of this concept the FQHE of electrons can be physically understood as a manifestation of the IQHE of CFs [11]. The statistics of anyons was determined in Refs. [10,12]. It was established that the wave function of the system changes by a complex phase factor exp[i$\pi\alpha$], when the quasiparticles are interchanged. For bosons $\alpha = 0$, for fermions $\alpha = 1$ and for anyons with $e^* = -e/3$ their statistical charge is $\alpha = -\frac{1}{3}$. As was shown in Ref. [13], there were no soft branches of neutral excitations in IQL. The energy gap Δ for formation of a quasielectron–quasihole pair has the scale of Coulomb energy $E_Q = e^2/\varepsilon l$, where ε is the dielectric constant of the background. However Δ was found to be small $\Delta \simeq 0.1E_Q$.The lowest branch was called as magnetoroton [13] and can be modeled as a quasiexciton [5]. As was mentioned in Ref. [5] the traditional methods and concepts based either on the neglecting of the electron–electron interaction or on self-consistent approximation are inapplicable to IQL. In a strong magnetic field the binding energy of an exciton increases from R_y to I_l. Small parameter of the theory, as in the present paper, is the ratio $I_l/\hbar\omega_c < 1$. Because the filling factor ν basically determines the underlying physics, it cannot be changed arbitrarily, and cannot serve as a small parameter in the case of FQHE. Exact numerical diagonalization for a few number of particles $N \leqslant 10$ proved to be the most powerful tool in studies of such systems [5]. The spherical geometry for these calculations was proposed [10,14], considering a few number of particles on the surface of a sphere with the radius $R = \sqrt{S}l$, so as the density of the particles on thesphere to be equal with the filling factor of 2DEG. The magnetic monopole in the center of the sphere creates a magnetic flux through the sphere $2S\Phi_0$, which is multiple to the flux quantum $\Phi_0 = 2\pi\hbar c/e$. The angular momentum L of a quantum state on the sphere and the quasimomentum k of the FQHE state on the plane obey the relation $L = Rk$. Spherical model is characterized by continuous rotational group, which is analogous with the continuous translational symmetry in the plane.

The experimental probe and the revealing of the physical properties of the 2DEG are based not only on the magnetoresistence measurements, but also on the photoluminescence (PL) methods. As was mentioned in Ref. [15] the connection of PL anomalies with the microscopic properties of IQL has been studied theoretically for over a decade. The role of photoinjected hole is determined by the influence of its Coulomb potential on the 2DEG. A new electron state, which can appear in such conditions were studied theoretically [5,15–21]. Experimentally observed doublets were attributed to new states mentioned below. For the values of the ratio h/l, where h is the distance between the plane occupied by the 2DEG and the plane, where the solitary hole is injected, which serves as a parameter of the theory, such that $h/l \ll 1$ the perturbative field of the hole is strong, the electron density in the vicinity of the hole strongly deviate from its mean value [5,15]. It leads to strong renormalization of the exciton properties. The numerical simulations for few electron system are accessible. For the opposite limit case of strongly asymmetric system $h/l \gg 1$ the approach based on the anyon concept seems to be more promising [5]. In the former case the giant suppression of the exciton dispersion law, the creation of the exciton-polaron state, where the role of phonons is played by magnetorotons, as well as the possible creation of the bound state including slow magnetoexciton and magnetoroton were established [16,17]. In the later case a concept of a new type of exciton, namely anyon exciton, was proposed [18,19]. It consists from a hole and of q quasielectrons, each of them being charged by a fractional charge $-e/q$. At $h/l \gg 1$ the mean separation between anyons in an exciton is about h and is larger than the anyon size l. The anyons in this case are well defined quasiparticles and the anyon exciton is an atom-like entity having internal degrees of freedom. Its energy spectrum would comprise a multiplicity of branches [5,18,19]. At present time different variants of the neutral and charged anyon excitons have been studied, but the understanding of all experimentally discovered and reported anomalies is not complete [15]. In the papers [15,20,21] the ideas about the trions immersed in the IQLs, which are stable and could explain the discontinuities of the PL spectra are discussed. As was mentioned in Ref. [15] the many-body IQL dynamics adds to few-body excitonic effects the own complexity. It was argued [15] that trions remain stable in realistic dopped wells. In analogy with the conventional excitons and trions in the papers [15,20,21] the different anyon variants were investigated in comparison with the previous papers [16–19]. Some types of anyon trions such as singlet, triplet dark and triplet bright have been discussed so far.

Properties of the symmetric 2D electron–hole (e–h) system (i.e. $h = 0$), with equal concentrations for both components, with coincident matrix elements of Coulomb electron–electron, hole–hole and electron–hole interactions in a strong perpendicular magnetic field also attracted a great attention during last two decades [22–29]. A hidden symmetry and the multiplicative states were discussed in many papers [16,20,26]. The collective states such as the Bose–Einstein condensation (BEC) of 2D magnetoexcitons and the formation of metallic-type electron–hole liquid (EHL) were investigated in Refs. [22–29]. The search for Bose–Einstein condensates has become a milestone in the condensed matter physics [30]. The remarkable properties of superfluids and superconductors are intimately related to the existence of a bosonic condensate of composite particles consisting of an even number of fermions. In highly excited semiconductors the role of such composite bosons is taken on by excitons, which are bound states of electrons and holes. Furthermore, the excitonic system has been viewed as a keystone system for exploration of the BEC phenomena, since it allows to control particle densities and interactions *in situ*. Promising candidates for experimental realization of such system are semiconductor quantum wells (QWs) [31], which have a number of advantages compared to the bulk systems. The coherent pairing of electrons and holes occupying only the lowes Landau levels (LLLs) was studied using the Keldysh–Kozlov–Kopaev method and the generalized random-phase approximation (RPA) [27]. The BEC of magnetoexcitons takes place in a single exciton state with wave vector k, supposing that the high density of electrons in the conduction band and of holes in the valence band were created in a single QW structure with size quantization much greater than the Landau quantization. In the case $k\neq 0$ a new metastable dielectric liquid phase formed by Bose–Einstein condensed magnetoexcitons was revealed [27,28]. The importance of the excited Landau levels (ELLs) and their influence on the ground states of the systems was first noticed by the authors of the papers [23–26]. The influence of the first excited Landau levels (FELLs) of electrons and holes was discussed in details in paper [28]. The indirect attraction between electrons (e–e), between holes (h–h) and between electrons and holes (e–h) due to the virtual simultaneous quantum transitions of the interacting charges from LLLs to the FELLs is a result of their Coulomb scattering. The first step of the scattering and the return back to the initial states were described in the second order of the perturbation theory. The purpose of the present paper is the detailed study of the influence of virtual quantum transitions of the Coulomb interacting particles from the LLLs to the ELLs. As in the case of FELLs, the influence of the another ELLs gives rise to the Hartree terms of the overall attractive indirect interaction between the particles. At the same time the Fock terms and the Bogoliubov u-v transformation terms of this indirect interaction are very important in some region of the condensate wave vectors and cannot be generated by another sources. Their search justifies the made investigation. We consider two approaches of the problem. One of them could be named as e–h approach, and another one as excitonic description. But this division is only conditional because the exciton creation and annihilation operators are constructed in their turn from the electron and hole operators. Nevertheless two approaches permit to elucidate different aspects of the problem and we will use both of them. In the first direction we will consider the electrons and holes lying on their LLLs and interacting between them by the forces, which are modified by the ELLS. On this base we will obtain two main results concerning the influence of ELLs on two phases formed by e–h system, namely on the BEC-ed phase and on EHL phase. The main results in this direction were published in summary form in our previous paper [29]. In the present paper we add many supplementary explications, derivations of the more important results and their illustrations. In the excitonic approach we will pay attention to the influence of ELLs on the wave functions and on the energy levels of the lowest magnetoexciton band, which permits to determine more exact expressions for the exciton creation and annihilation operators. They play a key role in the elaboration of the adequate theory of BEC. These operators and the corresponding wave functions represent the superpositions of the states involving different ELLs. The theory of the BEC of 2D magnetoexcitons on the superposition state is proposed.

The paper is organized as follows. In Section 2 the matrix elements of the Coulomb scattering processes were determined including the simultaneous excitations of two quasiparticles from the LLLs to ELLs. In Section 3 the influence of the ELLs on the ground state energies of the condensed magnetoexcitons and of the EHL were studied in the Hartree–Fock (HF) approximation. Section 4 is devoted to excitonic approach. The wave function, energy spectrum and exciton creation and annihilation operators were determined. On their base the description of the BEC of 2D magnetoexcitons in the superposition state is proposed. The conclusions and summary are given in Section 5.

2. Matrix elements of the Coulomb interaction: simultaneous quantum transitions

We consider a 2D electron–hole system in a perpendicular magnetic field, which is assumed to be strong enough such that LL quantization $\hbar\omega_c$ is larger than exciton binding energy, while the magnetic length l is smaller than 2D exciton Bohr radius a_{ex}^{2D}. Taking the magnetic field in the Landau gauge $\mathbf{A} = (-Hy, 0, 0)$, the electron states can be obtained straightforwardly by means of a mixed basis set of Landau functions and plane waves:

$$\psi^{e}_{n,p}(x,y) = \frac{1}{\sqrt{L_x l\sqrt{\pi}}} e^{ipx} \exp\left[-\frac{(y-pl^2)^2}{2l^2}\right] H_n\left(\frac{y-pl^2}{l}\right), \tag{1}$$

where L_x is a lateral size of 2D layer, n is the principal quantum number—Landau level index, and H_n is Hermite polynomial. Thus, the wave function is a plane wave in x-direction and exponentially localized in y-direction with the center at pl^2 and localization length of the order of l. Note, that one-dimensional (1D) wave vector p is an eigenvalue of the magnetic momentum operator [22]. For a charge neutral systems, like exciton, the magnetic momentum is a conserved quantity playing role of the total center-of-mass momentum. Therefore the total momentum in x-direction is a good quantum number and will be used for labeling eigenstates.

The hole states $\psi^{h}_{n,p}(x,y)$ can be obtained from the Eq. (1) by conserving p in the plane wave and changing its sign to $-p$ in the Landau part of electron wave function, since it has an opposite electric charge and is pulled by the Lorentz force in opposite direction along the y-axis. The energies of electron and hole accounted from the corresponding LLLs energies are simply $n_e\hbar\omega_{ce}$ and $n_h\hbar\omega_{ch}$, respectively. Hereafter we use a subscript notation to distinguish electrons and holes, unless otherwise stated. Under this conditions the Hamiltonian of the spin-polarized system can be written as

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{Coul}}, \tag{2}$$

with $\hat{H}_0$ term being the Hamiltonian of non-interacting electrons and holes

$$\hat{H}_0 = \sum_{n=0}^{\infty}\sum_{p}(n\hbar\omega_{ce} - \mu_e)a^{\dagger}_{n,p}a_{n,p} + \sum_{n=0}^{\infty}\sum_{q}(n\hbar\omega_{ch} - \mu_h)b^{\dagger}_{n,q}b_{n,q}. \tag{3}$$

The operators $a^{\dagger}_{n,p}$ and $a_{n,p}$ stand for creation and annihilation of an electron in the state with a momentum p and LL index n; $b^{\dagger}_{n,q}$ and $b_{n,q}$ hold the same meaning for hole; μ_e and μ_h are chemical potentials of electrons and holes. The number of discrete states is $N = L_x\Delta p/2\pi = S/2\pi l^2$, where S is the layer area.

The interaction term is given by $\hat{H}_{\text{Coul}}$:

$$\begin{aligned}\hat{H}_{\text{Coul}} = \frac{1}{2}\sum_{p,q,s}\sum_{n,m,n',m'} &[F_{e-e}(p,n;q,m;p-s,n';q+s,m')\\ &\times a^{\dagger}_{n,p}a^{\dagger}_{m,q}a_{m',q+s}a_{n',p-s}\\ &+ F_{h-h}(p,n;q,m;p-s,n';q+s,m')\\ &\times b^{\dagger}_{n,p}b^{\dagger}_{m,q}b_{m',q+s}b_{n',p-s}]\\ &-\sum_{p,q,s}\sum_{n,m,n',m'} F_{e-h}\\ &\times(p,n;q,m;p-s,n';q+s,m')\\ &\times a^{\dagger}_{n,p}b^{\dagger}_{m,q}b_{m',q+s}a_{n',p-s},\end{aligned} \tag{4}$$

with Coulomb matrix elements

$$\begin{aligned}&F_{i-j}(p,n;q,m;p-s,n';q+s,m')\\ &= \int\int \mathrm{d}\varrho_1\mathrm{d}\varrho_2\psi^{i*}_{n,p}(\varrho_1)\psi^{j*}_{m,q}(\rho_2)\frac{e^2}{\varepsilon_0|\varrho_1-\varrho_2|}\\ &\times\psi^{i}_{n',p-s}(\varrho_1)\psi^{j}_{m',q+s}(\varrho_2), \quad i,j = e,h.\end{aligned} \tag{5}$$

Here the integration of envelope parts of the wave functions Eq. (1) was performed. The periodic parts of the full Bloch functions being integrated on the elementary lattice cell were excluded from the final expression. The exchange terms related with the quantum transitions from the valence band to conduction band and giving rise to ortho–para exciton splitting and to longitudinal-transverse splitting of the states of dipole active excitons are not taken into account. The variables ϱ_i are 2D vectors and ε_0 is the background dielectric constant.

The matrix elements of the Coulomb interaction involving only the LLLs, i.e. $n = m = n' = m' = 0$ contribution to the interaction Hamiltonian, labeled as $H^{\text{LLL}}_{\text{Coul}}$ were studied in Refs. [25–28]. As an example we will demonstrate some of 48 matrix elements with quantum numbers $n, m, n', m' = 0, 1$, which involve the first two LLs, namely the LLLs and FELLs of electrons and holes:

$$\begin{aligned}F_{i-i}(p,0;q,0;p-s,0;q+s,0) &= \sum_{\kappa} W_{s,\kappa}f(\kappa,p-q-s),\\ F_{i-i}(p,1;q,0;p-s,0;q+s,0) &= \sum_{\kappa} W_{s,\kappa}\frac{(s-\mathrm{i}\kappa)l}{\sqrt{2}}\\ &\quad\times f(\kappa,p-q-s),\\ F_{i-i}(p,1;q,1;p-s,0;q+s,0) &= -\sum_{\kappa} W_{s,\kappa}\frac{(\mathrm{i}\kappa-s)^2l^2}{2}\\ &\quad\times f(\kappa,p-q-s),\\ F_{i-i}(p,1;q,1;p-s,1;q+s,0) &= \sum_{\kappa} W_{s,\kappa}\frac{(s-\mathrm{i}\kappa)l}{\sqrt{2}}\\ &\quad\times\left[1-\frac{(s^2+\kappa^2)l^2}{2}\right]\\ &\quad\times f(\kappa,p-q-s),\end{aligned}$$

$$\begin{aligned}&F_{i-i}(p,1;q,1;p-s,1;q+s,1)\\ &\quad= \sum_{\kappa} W_{s,\kappa}\left[1-\frac{(s^2+\kappa^2)l^2}{2}\right]^2\times f(\kappa,p-q-s).\end{aligned} \tag{6}$$

The similar e–h matrix elements are

$$\begin{aligned}&F_{e-h}(p,0;q,0;p-s,0;q+s,0)\\ &\quad= \sum_{\kappa} W_{s,\kappa}f(\kappa,p+q),\\ &F_{e-h}(p,1;q,0;p-s,0;q+s,0)\\ &\quad= \sum_{\kappa} W_{s,\kappa}\frac{(\mathrm{i}\kappa-s)l}{\sqrt{2}}f(\kappa,p+q),\\ &F_{e-h}(p,1;q,1;p-s,0;q+s,0)\\ &\quad= \sum_{\kappa} W_{s,\kappa}\frac{(s^2+\kappa^2)l^2}{2}f(\kappa,p+q),\\ &F_{e-h}(p,1;q,1;p-s,1;q+s,0)\\ &\quad= \sum_{\kappa} W_{s,\kappa}\frac{(\mathrm{i}\kappa+s)l}{\sqrt{2}}\left[1-\frac{(s^2+\kappa^2)l^2}{2}\right]\\ &\quad\times f(\kappa,p+q),\\ &F_{e-h}(p,1;q,1;p-s,1;q+s,1)\\ &\quad= \sum_{\kappa} W_{s,\kappa}\left[1-\frac{(s^2+\kappa^2)l^2}{2}\right]^2\times f(\kappa,p+q).\end{aligned} \tag{7}$$

Here we used the notations

$$W_{s,\kappa} = V_{s,\kappa}\exp\left[-\frac{(s^2+\kappa^2)l^2}{2}\right], \quad f(\kappa,p) = \exp(\mathrm{i}\kappa p l^2), \tag{8}$$

where $V_{s,\kappa}$ is the 2D Fourier transform of the Coulomb interaction

$$V_{s,\kappa} = \frac{2\pi e^2}{\varepsilon_0 S\sqrt{s^2+\kappa^2}}. \tag{9}$$

The application of these matrix elements for derivation of the lowest exciton bands will be shown in Section 4. The main subject of the present paper concerns the simultaneous quantum transitions of two quasiparticles due to their Coulomb scattering.

Quantum transitions associated with Coulomb interactions are allowed without changing the spins of interacting particles. Since we are concerned only in the LLLs, which can accommodate all particles at zero temperature and high magnetic field, we consider virtual transitions, where a particle is first promoted from the LLL to the ELL, and then reverted to the LLL. We restrict ourselves with the simultaneous virtual transitions of two particles, but allowing them to be excited in LLs with arbitrary n and m. Such virtual transitions correspond to matrix elements $F_{i-j}(p,0;q,0;p-s,n;q+s,m)$ and $F_{i-j}(p,n;q,m;p-s,0;q+s,0)$ with $i,j=e,h$. As a result, these transitions will induce indirect interaction between particles in the LLLs and influence essentially on the BEC of magnetoexcitons. Such indirect interaction is attractive and appears in the second order of the perturbation theory. Following the statements of the paper [26], the main role is played by the simultaneous quantum transitions with $n'=m'=n$. However, the quantum transitions with $n\neq m$ have to be also taken into account and we will show that the virtual quantum transitions with participation of the e–h pair give rise to the contributions of two types, which are both depended on the magnetoexciton wave vector $\mathbf{k}$, but exhibitingvanishing or non-vanishing behavior at the point $k=0$. The indirect e–e and h–h interaction lead to contributions, which do not depend on $\mathbf{k}$. It was shown [28] that for $n=m=1$ this indirect interaction gives rise to the shift of the magnetoexciton levels and influence on BEC. The aim of this section is to generalize the results obtained in Ref. [28] and to determine the influence of all ELLs with arbitrary n and m.

We start with rewriting the Hamiltonian Eq. (2) by separating the term $H^{\mathrm{LLL}}_{\mathrm{Coul}}$, which contains only the LLLs and the term $H^{\mathrm{ELL}}_{\mathrm{Coul}}$, which describes the simultaneous transitions $(0,0)\rightleftarrows(n,m)$ discussed above, while all others terms entering Eq. (2) will be neglected:

$$H = H_0 + H^{\mathrm{LLL}}_{\mathrm{Coul}} + H^{\mathrm{ELL}}_{\mathrm{Coul}}. \tag{10}$$

From now on particle operators with $n=m=0$ will be denoted as $a_p^\dagger, a_p, b_p^\dagger$ and b_p. The term $H^{\mathrm{ELL}}_{\mathrm{Coul}}$ can be excluded from the Hamiltonian Eq. (10) with the aid of unitary transformation [32,33] $\hat{U}=\exp[\mathrm{i}\hat{S}]$, where $\hat{S}=\hat{S}^\dagger$ is determined from the equation

$$\mathrm{i}[\hat{H}_0,\hat{S}] + H^{\mathrm{ELL}}_{\mathrm{Coul}} = 0. \tag{11}$$

Averaging the transformed Hamiltonian on the ground state of electrons and holes in ELLs $|0\rangle_{\mathrm{ELL}}$ we obtain an effective Hamiltonian

$$\begin{aligned} H_{\mathrm{eff}} &= {}_{\mathrm{ELL}}\langle 0|\mathrm{e}^{-\mathrm{i}S}\hat{H}\mathrm{e}^{\mathrm{i}S}|0\rangle_{\mathrm{ELL}} \\ &\simeq -\mu_{\mathrm{e}}\sum_p a_p^\dagger a_p - \mu_{\mathrm{h}}\sum_p b_p^\dagger b_p + H^{\mathrm{LLL}}_{\mathrm{Coul}} \\ &\quad + \frac{\mathrm{i}}{2}\,{}_{\mathrm{ELL}}\langle 0|[H^{\mathrm{ELL}}_{\mathrm{Coul}},\hat{S}]|0\rangle_{\mathrm{ELL}}, \end{aligned} \tag{12}$$

which can be written as

$$\begin{aligned} H_{\mathrm{eff}} &= -\mu_{\mathrm{e}}\sum_p a_p^\dagger a_p - \mu_{\mathrm{h}}\sum_p b_p^\dagger b_p + H^{\mathrm{LLL}}_{\mathrm{Coul}} \\ &\quad -\frac{1}{2}\sum_{p,q,z}\phi_{e-e}(p,q,z)a_p^\dagger a_q^\dagger a_{q+z}a_{p-z} \\ &\quad -\frac{1}{2}\sum_{p,q,z}\phi_{h-h}(p,q,z)b_p^\dagger b_q^\dagger b_{q+z}b_{p-z} \\ &\quad -\sum_{p,q,z}\phi_{e-h}(p,q,z)a_p^\dagger b_q^\dagger b_{q+z}a_{p-z}. \end{aligned} \tag{13}$$

Here the indirect interaction matrix elements $\phi_{i-j}(p,q,z)$ are given by the expressions

$$\begin{aligned} \phi_{i-j}(p,q,z) &= \sum_{n,m}\frac{\phi_{i-j}(p,q,z;n,m)}{n\hbar\omega_{\mathrm{c}i}+m\hbar\omega_{\mathrm{c}j}}, \\ \phi_{i-j}(p,q,z;n,m) &= \sum_t F_{i-j}(p,0;q,0;p-t,n;q+t,m) \\ &\quad \times F_{i-j}(p-t,n;q+t,m;\quad p-z,0; \\ &\qquad q+z,0). \end{aligned} \tag{14}$$

Making use of the definition Eq. (5) and notations of Eq. (8) one can write Coulomb matrix elements in the following form:

$$\begin{aligned} &F_{i-i}(p,n;q,m;p-s,0;q+s,0) \\ &\quad = \frac{(-1)^m}{\sqrt{2^{n+m}n!m!}}\sum_\kappa W_{s,\kappa}f(\kappa,p-q-s)(-s+\mathrm{i}\kappa)^{n+m}l^{n+m}, \\ &F_{i-i}(p,0;q,0;p-s,n;q+s,m) \\ &\quad = \frac{(-1)^m}{\sqrt{2^{n+m}n!m!}}\sum_\kappa W_{s,\kappa}f(\kappa,p-q-s)(s+\mathrm{i}\kappa)^{n+m}l^{n+m}, \\ &F_{e-h}(p,0;q,0;p-t,n;q+t,m) \\ &\quad = \frac{1}{\sqrt{2^{n+m}n!m!}}\sum_\kappa W_{t,\kappa}f(\kappa,p+q)[t+\mathrm{i}\kappa]^n[t-\mathrm{i}\kappa]^m l^{n+m}, \\ &F_{e-h}(p-t,n;q+t,m;p-z,0;q+z,0) \\ &\quad = \frac{1}{\sqrt{2^{n+m}n!m!}}\sum_\sigma W_{t-z,\sigma}f(\sigma,p+q)[(t-z)+\mathrm{i}\sigma]^n \\ &\quad \times[(t-z)-\mathrm{i}\sigma]^m l^{n+m}. \end{aligned} \tag{15}$$

After straightforward calculation the indirect interaction matrix elements take the following form:

$$\begin{aligned} \phi_{i-i}(p,q,z,n,m) &= \frac{l^{2(n+m)}}{2^{n+m}n!m!}\sum_{t,\kappa,\sigma}W_{t,\kappa}W_{z-t,\sigma}f(\kappa,p-q-t) \\ &\quad \times f(\kappa,p-q-t-z)(t+\mathrm{i}\kappa)^{n+m} \\ &\quad \times(t-z+\mathrm{i}\sigma)^{n+m}, \end{aligned}$$

$$\phi_{e-h}(p,q,z;n,m) = \frac{l^{2(n+m)}}{2^{n+m}n!m!}\sum_{t,\kappa,\sigma} W_{t,\kappa}W_{t-z,\sigma}f(\kappa+\sigma,p+q) \times(t+\mathrm{i}\kappa)^n(t-\mathrm{i}\kappa)^m[(t-z)+\mathrm{i}\sigma]^n \times[(t-z)-\mathrm{i}\sigma]^m. \quad (16)$$

A closed form for these expressions cannot be found, however, in special cases the sums of these matrix elements can be reduced to much simpler forms. For example, in the particular case of electrons and holes bound to excitons with a conserved wave vector $k_x = p+q$ the influence of indirect interaction between electrons and holes leads to the exciton level energy shift $\Delta(k)$ and is expressed by the sum

$$\Delta(k) = \sum_z \phi_{e-h}(p,k_x-p,z)\exp(-\mathrm{i}k_y z l^2) = \frac{I_l^2\exp(-k^2l^2)}{\pi}\sum_{n,m}\frac{(kl)^{2|n-m|}}{n!m!2^{|n-m|}(n\hbar\omega_{ce}+m\hbar\omega_{ch})} \times\left[\frac{\Gamma((m+n+|n-m|+1)/2)}{\Gamma(|n-m|+1)}\,{}_1F_1 \times\left(\frac{|n-m|+1-(n+m)}{2};|n-m|+1;\frac{(kl)^2}{2}\right)\right]^2, \quad (17)$$

and for virtual transitions of an electron and a hole to LLs with the same index $n=m$ it reduces to

$$\Delta'(k) = \frac{I_l^2\exp(-k^2l^2)}{\pi(\hbar\omega_{ce}+\hbar\omega_{ch})}\sum_{n\geqslant 1}\frac{[\Gamma(n+1/2)]^2}{(n!)^2 n}\left[{}_1F_1 \times\left(-n+\frac{1}{2};1;\frac{(kl)^2}{2}\right)\right]^2, \quad (18)$$

with $\Gamma(z)$ being the complete Gamma function and ${}_1F_1(a;b;z)$ is the confluent hypergeometric function [34]; $I_l = \sqrt{\pi/2}\,e^2/\varepsilon_0 l$ is magnetoexciton binding energy. At the bottom of the lowest magnetoexciton band the energy shift equals to

$$\Delta(0) = \Delta'(0) = \frac{I_l^2}{\pi(\hbar\omega_{ce}+\hbar\omega_{ch})}\sum_{n\geqslant 1}\frac{[\Gamma(n+1/2)]^2}{n(n!)^2} = \frac{4I_l^2}{\pi(\hbar\omega_{ce}+\hbar\omega_{ch})}(\pi\ln 2 - 2G) = \frac{4I_l^2}{\pi(\hbar\omega_{ce}+\hbar\omega_{ch})}0.344, \quad (19)$$

where G is Catalan's constant ($\simeq 0.915966$). Note that at $k=0$ the contribution of virtual transitions with $n\neq m$ vanishes and only transitions with the same index n contribute to the indirect interaction between electron and hole. But for finite values of k one has to account for virtual transitions with $n\neq m$.

In addition, we calculate the sum of the diagonal matrix elements of indirect e–h interaction

$$A_{e-h} = \sum_q \phi_{e-h}(p,q,0) = \frac{I_l^2}{\pi}\sum_{n\geqslant 1}\sum_{m\geqslant 1}\frac{(n+m-1)!}{2^{n+m}n!m!(n\hbar\omega_{ce}+m\hbar\omega_{ch})}, \quad (20)$$

which can be rewritten for the case $\omega_{ce}=\omega_{ch}=\omega_c$ (hereafter we assume equal cyclotron frequencies for electron and hole) as

$$A_{e-h} = \frac{I_l^2}{\pi\hbar\omega_c}\sum_{n\geqslant 1}\sum_{m\geqslant 1}\frac{(n+m-1)!}{2^{n+m}n!m!(n+m)} = \frac{I_l^2}{\pi\hbar\omega_c}S, \quad (21)$$

where

$$S = \frac{1}{2}\ln^2 2 - \frac{\pi^2}{12} + 1.06269 \approx 0.481. \quad (22)$$

Besides the indirect coupling of e–h pairs we consider indirect e–e and h–h interactions expressed by the matrix elements ϕ_{e-e} and ϕ_{h-h}. It can be seen that the sum of diagonal matrix elements determines the Hartree contribution of the indirect interaction and is equal to that one obtained for e–h indirect interaction:

$$A_{i-i} = \sum_q \phi_{i-i}(p,q,0) = \frac{I_l^2}{\pi\hbar\omega_c}\sum_{n\geqslant 1}\sum_{m\geqslant 1}\frac{(n+m-1)!}{2^{n+m}n!m!(n+m)} = \frac{I_l^2}{\pi\hbar\omega_c}S. \quad (23)$$

The sum of the non-diagonal matrix elements $\phi_{i-i}(p,p-z,z)$ will determine the exchange contribution of the indirect supplementary interaction as follows:

$$B_{i-i} = \frac{2I_l^2}{\pi\hbar\omega_c}\sum_{n,m}\frac{1}{2^{n+m}n!m!(n+m)}\int_0^\infty \mathrm{d}x\mathrm{e}^{-x^2/2}x^{n+m} \times\int_0^\infty \mathrm{d}y\mathrm{e}^{-y^2/2}y^{n+m}J_{n+m}(xy) = \frac{2I_l^2}{\pi\hbar\omega_c}T, \quad (24)$$

where $J_n(z)$ is Bessel function of the first kind. After a thorough analysis of the sum in the above expression it can be shown that $T \simeq 0.2161$.

3. Influence of ELLs on the collective states of 2D electron–hole system

The supplementary indirect interaction entering the Hamiltonian Eq. (13) influences on the collective properties of 2D e–h system. We will consider two alternative states: namely, the BEC of magnetoexcitons and the metallic-type EHL. Both of them will be described in HF approximation. The indirect interactions between particles in the e–h system differ from their Coulomb interaction. The average values of the Coulomb interactions in the HF approximation give rise to direct-pairing or Hartree terms and to exchange-pairing or Fock terms. The direct-pairing terms of the e–e, h–h and e–h Coulomb interactions cancel each other due the condition of electro-neutrality of the e–h system, whereas the Coulomb exchange e–e and h–h interactions are negative. They give rise to the attraction in the system and facilitate the creation of Coulomb correlated e–h plasma and e–h liquid. The supplementary attraction in the system increases the binding energy of magnetoexcitons and at the same time lowers the energy per one pair in the composition of EHL. These competing processes will be compared. On the contrary, the exchange pairing terms of the supplementary interaction are positive, giving rise to the repulsion in the e–h system. They act in the favor of the BEC of magnetoexcitons tending to stabilize it and partially diminish the binding energy of EHL. First, we will discuss the BEC of magnetoexcitons in HFBA.

The ground state energy E_g of the Bose–Einstein condensed magnetoexcitons in Ref. [27] was calculated beyond the HFBA on the base of Pauli–Feynman theorem following the proposal of Comte and Nozieres [35,36]. Being applied to 2D magnetoexcitons the main formula reads as

$$E_g = -N_{ex}\sum_{\mathbf{Q}} W_{\mathbf{Q}} - \sum_{\mathbf{Q}} \int_0^{\infty} \frac{\hbar d\omega}{2\pi} \int_0^{e^2} \frac{d\lambda}{\lambda} \mathrm{Im}\left(\frac{1}{\varepsilon(\mathbf{Q},\omega,\lambda)}\right). \tag{25}$$

Here N_{ex} is the average number of magnetoexcitons and $\varepsilon(\mathbf{Q},\omega,\lambda)$ is their dielectric constant in the case of hypothetical e–h system with square electric charge equal to λ. Substituting in Eq. (25) the dielectric constant in HFA called as $\varepsilon^{HF}(\mathbf{Q},\omega,\lambda)$ or in RPA denoted as $\varepsilon^{RPA}(\mathbf{Q},\omega,\lambda)$ in Ref. [27] the results were obtained in HFBA or beyond it with account for correlation energy.

Two approximations for dielectric constant have different dependencies on the polarizability $4\pi\alpha_0^{HF}(\mathbf{Q},\omega,\lambda)$

$$\varepsilon^{RPA}(\mathbf{Q},\omega,\lambda) = 1 + 4\pi\alpha_0^{HF}(\mathbf{Q},\omega,\lambda),$$
$$\frac{1}{\varepsilon^{HF}(\mathbf{Q},\omega,\lambda)} = 1 - 4\pi\alpha_0^{HF}(\mathbf{Q},\omega,\lambda). \tag{26}$$

The polarizability can be calculated in the approximation of a weak response, if the wave function of the system in zero order approximation is known. In the case of BEC of magnetoexcitons as a ground state wave function was chosen the BCS-type wave function [27] $|\psi_g(k)\rangle$ and as the excited wave functions the wave functions of the coherent excited states introduced in Ref. [36] for e–h systems in a similar way as it was done by Anderson [37] in the theory of superconductors. The ground state wave function was introduced following Keldysh–Kozlov method [38] by the action of the displacement unitary transformation $\hat{D}(\sqrt{N_{ex}})$ on the vacuum state of the initially introduced electron–hole operators

$$|\psi_g(k)\rangle = \hat{D}(\sqrt{N_{ex}})|0\rangle, \quad a_p|0\rangle = b_p|0\rangle = 0. \tag{27}$$

The coherent excited states were generated as follows [27]:

$$\left|\psi^e\left(q \pm \frac{Q_x}{2}\right)\right\rangle = a^{\dagger}_{q+Q_x/2} a_{q-Q_x/2}|\psi_g(k)\rangle. \tag{28}$$

The unitary transformation $\hat{D}(\sqrt{N_{ex}})$ breaks the gauge symmetry of the initial Hamiltonian Eq. (13) transforming it to a new Hamiltonian $\hat{D}H_{eff}\hat{D}^{\dagger}$, yielding the ground state wave function Eq. (27) and macroscopic displacement $\sqrt{N_{ex}}$ of the exciton creation operator

$$d^{\dagger}(k) = \frac{1}{\sqrt{N}}\sum_t e^{-iQ_y t l^2} a^{\dagger}_{k_x/2+t} b^{\dagger}_{k_x/2-t}. \tag{29}$$

Note that contrary to the Glauber coherent states [39] the exciton creation and annihilation operators are not pure Bose operators but only quasi-boson operators [40].

The unitary transformation $\hat{D}(\sqrt{N_{ex}}) = \exp[\sqrt{N_{ex}}(d^{\dagger}(k) - d(k))]$ of the Hamiltonian implies the unitary transformations of the operators

$$\hat{D}a_p\hat{D}^{\dagger} \equiv \alpha_p = ua_p - v\left(p - \frac{k_x}{2}\right)b^{+}_{k_x-p},$$
$$\hat{D}b_p\hat{D}^{\dagger} \equiv \beta_p = ub_p + v\left(\frac{k_x}{2} - p\right)a^{+}_{k_x-p}, \tag{30}$$

yielding inverse transformation

$$a_p = u\alpha_p + v\left(p - \frac{k_x}{2}\right)\beta^{\dagger}_{k_x-p}, \quad b_p = u\beta_p - v\left(\frac{k_x}{2} - p\right)\alpha^{\dagger}_{k_x-p}, \tag{31}$$

with the coefficients

$$v(t) = ve^{-ik_y t l^2}, \quad v = \sin(\sqrt{2\pi l^2 n_{ex}}), \quad u = \cos(\sqrt{2\pi l^2 n_{ex}}),$$
$$n_{ex} = N_{ex}/S.$$

The restriction of the LLL implies the following equalities [27]:

$$v^2 = N_{ex}/N, \quad n_{ex} = \frac{v^2}{2\pi l^2},$$

where v^2 is the filling factor of the LLL. The last line immediately brings us to the relations $u = \cos v$ and $v = \sin v$, which can be satisfied only in the limit $v < 1$. The theory developed in Ref. [27] and its application below has to be treated with the restriction $v < 1$. To avoid this constraint it is necessary to generalize the structure of the exciton creation operator Eq. (29) including in its composition the creation operators of electrons and holes at least in a few number of ELLs. This improvement will be discussed in the next section, where the FELL is included.

The Hamiltonian of Eq. (13) after the unitary transformation (31) will contain operators $\alpha_p^{\dagger}, \alpha_p, \beta_p^{\dagger}, \beta_p$ in arbitrary ordering. Their normal ordering will generate a constant U playing the role of the ground state energy of HFBA, a quadratic term H_2 similar with the quadratic expression Eq. (36) of Ref. [27], and a quartic term H'. Hence, the terms entering H' contain only normal ordered operators $\alpha_p^{\dagger}, \alpha_p, \beta_p^{\dagger}, \beta_p$ instead of $a_p^{\dagger}, a_p, b_p^{\dagger}, b_p$. The average value of H'

on the ground state wave function Eq. (27) equals zero even for the term proportional to v^4. Its contribution is non-zero only in higher orders of the perturbation theory, when the coherent excited states were used, as it was demonstrated in the Ref. [27], where the correlation energy contain a factor v^4u^4. The role of smallness parameter is played by the filling factor $v^2 = 2\pi l^2 n_{ex} < 1$. At a given magnetic field v^2 can be altered arbitrary within interval $(0, 1)$ by changing the exciton concentration n_{ex}, or the total number of excitons in the system N_{ex}. Contrary to the small parameter $r = I_l/\hbar\omega_c$ discussed above and related with the intensity of the magnetic field there exists another independent parameter n_{ex} of different origin, which in our consideration must be small as compared to $1/2\pi l^2$.

The quadratic Hamiltonian H_2 is given below for the case of electrons and holes with equal masses $m_e = m_h$, cyclotron frequencies $\omega_{ce} = \omega_{ch} = \omega_c$ and chemical potentials $\mu_e = \mu_h = \mu/2$:

$$H_2 = \sum_p [E(\mathbf{k}, v^2, \mu) + (B - 2A)v^2(1 - 2v^2) + 2v^2(1 - v^2)\Delta(k)](\alpha_p^\dagger\alpha_p + \beta_p^\dagger\beta_p) + \sum_p \left[uv\left(\frac{k_x}{2} - p\right)\beta_{k_x-p}\alpha_p + uv\left(p - \frac{k_x}{2}\right)\alpha_p^\dagger\beta_{k_x-p}^\dagger\right] \times\{-\psi(\mathbf{k}, v^2, \mu) + 2v^2(B - 2A + \Delta(k)) - \Delta(k)\}. \quad (32)$$

Following the notations of Eqs. (40) and (41) of Ref. [27] we have

$$E(\mathbf{k}, v^2, \mu) = 2v^2u^2 I_{ex}(k) + I_l(v^4 - v^2u^2) - \frac{\mu}{2}(u^2 - v^2),$$
$$\psi(\mathbf{k}, v^2, \mu) = 2v^2 I_l + I_{ex}(k)(1 - 2v^2) + \mu, \quad (33)$$

whereas the coefficients $\Delta(k)$, A and B are determined by Eqs. (17), (21) and (24), respectively. Putting to zero the last bracket in Eq. (32), i.e. compensating the dangerous diagrams describing the spontaneous creation and annihilation of e–h pairs in the vacuum state Eq. (27), one can obtain the chemical potential μ of the system in the HFBA:

$$\mu^{HFB} = -\tilde{I}_{ex}(k) + 2v^2(B - 2A + \tilde{I}_{ex}(k) - I_l) = -\tilde{I}_{ex}(k) + 2v^2(B - 2A + \Delta(k) - E(k)). \quad (34)$$

Here the renormalized ionization potential of magnetoexcitons $\tilde{I}_{ex}(k)$ containing the correction due to influence of all ELLs was introduced:

$$\tilde{I}_{ex}(k) = I_{ex}(k) + \Delta(k), \quad I_{ex}(k) = I_l - E(k),$$
$$E_{ex}(k) = -I_{ex}(k). \quad (35)$$

Introducing the value μ^{HFB} in the remainder part of the first line of Eq. (32), Hamiltonian H_2 will take the form

$$H_2 = \sum_p \frac{\tilde{I}_{ex}(k)}{2}(\alpha_p^+\alpha_p + \beta_p^+\beta_p). \quad (36)$$

This Hamiltonian describes the single-particle elementary excitations from a single-exciton state with wave vector $\mathbf{k}$ of the condensed magnetoexcitons. To extract from the condensate one pair of new quasiparticles the energy cost $\tilde{I}_{ex}(k)$ is equivalent to unbinding energy, i.e. the excitation energy for one quasiparticle equals to $\tilde{I}_{ex}(k)/2$. Notice that the chemical potential μ^{HFB} in the point $v^2 = 0$ coincides on the energy scale with the position of the renormalized magnetoexciton energy band

$$\tilde{E}_{ex}(k) = -\tilde{I}_{ex}(k),$$

while in the point $2v^2 = 1$ it equals to the value $-I_l + B - 2A$ and does not depend on k. The concentration corrections to μ^{HFB} are determined by the term

$$2v^2(B - 2A + \Delta(k) - E(k)). \quad (37)$$

The term $-E(k)$ appears in the frame of the LLLs and was obtained in the Refs. [26,27]. It determines the instability of the ground state within the HFBA, when the corrections due to ELL are neglected. The term $B - 2A$ appears in both phases, not only in the case of BEC of magnetoexciton, but also in the case of EHL. The term $-2A$ is related with the average Hartree terms of the supplementary e–e, h–h and e–h interactions, whereas the term B with the average exchange terms of the supplementary e–e and h–h interactions. The term $2v^2\Delta(k)$ is according to e–h interaction and Bogoliubov u-v transformation and is named as Bogoliubov self-energy term [41]. As it will be shown below it does not appear in the case of EHL.

The renormalized ionization potential in a dimensionless form is represented in Fig. 1, when the parameter r was taken equal to 1 and $\frac{1}{2}$. Note the value 1 is the maximal possible value because the theory is valid only for $r < 1$. The resulting influence of the ELLs on the chemical potential $\mu(k, v^2)$ calculated in the HFBA is determined by the coefficient $(B - 2A + \Delta(k))$ as was mentioned above. In the dimensionless form it is represented in Fig. 2 also for two different ratios $r = 0.5{:}\,1$. This influence, as well as the influence of FELLs discussed in the paper [28], is essential only in the range of small values of $kl < 0.5$, decreasing rapidly with the increasing of kl. The inset on this figure represents the coefficient $[B - 2A + \Delta(k) - E(k)]$, which reflects the influence of both the LLLs and of the ELLs. Thus, the main result obtained so far, namely the dependence of the chemical potential μ^{HFB} in HFBA versus the filling factor v^2 of the LLL at different values of the dimensionless wave vector $kl = 0, 0.5, 1.0, 3.6$ and for two different values of ratio $r = 0.5{:}\,1$ is presented in Fig. 3. One can see that the BEC of 2D magnetoexcitons with wave vector $kl < 0.5$ is stable in HFBA. As was realized in Ref. [27,28] at greater values $kl > 0.5$ the influence of coherent excited states [37] is important and leads to the appearance of the metastable dielectric liquid phase.

Now we consider the EHL formation in HFA. We start with an effective Hamiltonian (13), but without chemical potentials μ_e and μ_h, and calculate the ground state energy E_{EHL} of EHL at $T = 0$ when the average values of electrons and holes numbers on the LLLs are equal to

$$\langle a_p^\dagger a_p\rangle = \langle b_p^\dagger b_p\rangle = v^2. \quad (38)$$

Here v^2 is the filling factor of LLLs. Applying the Wick theorem, we obtained the ground state energy

$$E_{\mathrm{EHL}} = -N_{\mathrm{e-h}}[v^2 I_l + v^2(2A - B)], \quad N_{\mathrm{e-h}} = Nv^2, \tag{39}$$

and the energy per one e–h pair $\varepsilon_{\mathrm{EHL}}$ of EHL in units of I_l

$$\frac{\varepsilon_{\mathrm{EHL}}}{I_l} = -v^2\left(1 + \frac{2I_l}{\pi\hbar\omega_{\mathrm{c}}}(S - T)\right). \tag{40}$$

Taking into account the estimated values $S = 0.481$ and $T = 0.216$ we have

$$\frac{\varepsilon_{\mathrm{EHL}}}{I_l} = -(1 + 0.168r)v^2, \quad r = I_l/\hbar\omega_{\mathrm{c}}. \tag{41}$$

The minimal value is achieved at filling factor $v^2 = 1$, and it determines the energy per pair inside EHD equal to

$$\varepsilon_{\mathrm{EHD}} = -I_l(1 + 0.168r). \tag{42}$$

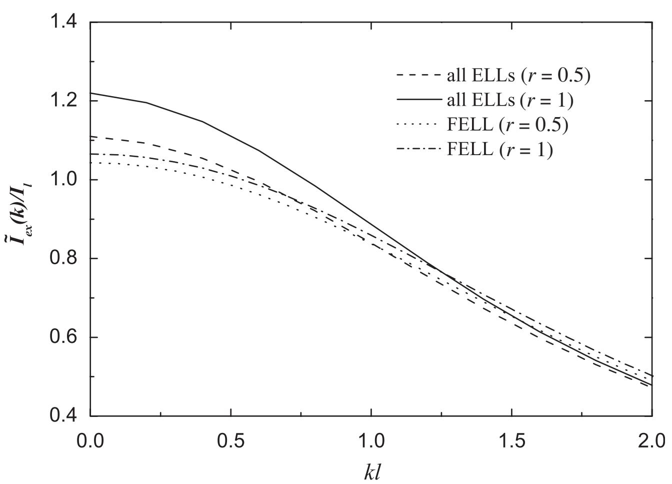

Fig. 1. Renormalized exciton ionization potential versus dimensionless wavevector kl for different values of the parameter $r = I_l/\hbar\omega_{\mathrm{c}}$. Solid line: $r = 1$; dashed line: $r = 0.5$. Dash-dotted line: ionization potential with the inclusion of only FELL for $r = 1$; dotted line: the same, but for $r = 0.5$ (see Section 4).

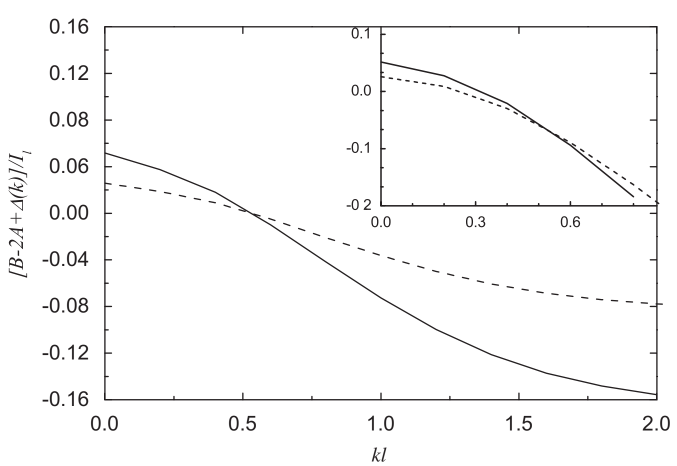

Fig. 2. Coefficient $B - 2A + \Delta(k)$ versus wavevector k for different values of the parameter $r = I_l/\hbar\omega_{\mathrm{c}}$. Solid line: $r = 1$; dashed line: $r = 0.5$. Inset: coefficient $B - 2A + \Delta(k) - E(k)$ versus wavevector k. Solid line: $r = 1$; dashed line: $r = 0.5$.

$\varepsilon_{\mathrm{EHL}}$ and $\varepsilon_{\mathrm{EHD}}$ depend on the ratio $r<1$. In spite of the restriction $r<1$ equivalent to a strong magnetic field condition, we will make also calculations in the case of maximal possible value $r = 1$. One can see that the corrections due to ELLs lower the energy per pair inside EHD by the value $0.168I_l$ for $r = 1$ and by the value $0.084I_l$ for $r = 0.5$. The energy per e–h pair of EHD is presented in Fig. 3 for different values of the ratio r. The energy $\varepsilon_{\mathrm{EHD}}$ is of the same order of magnitude as the chemical potential of condensed excitons with small wave vectors k and the coexistence of these two states is possible.

4. Excitonic approach: BEC in the superposition exciton state

The aim of this section is to generalize the results expressed by formulas (27)–(31), as well as to compare the

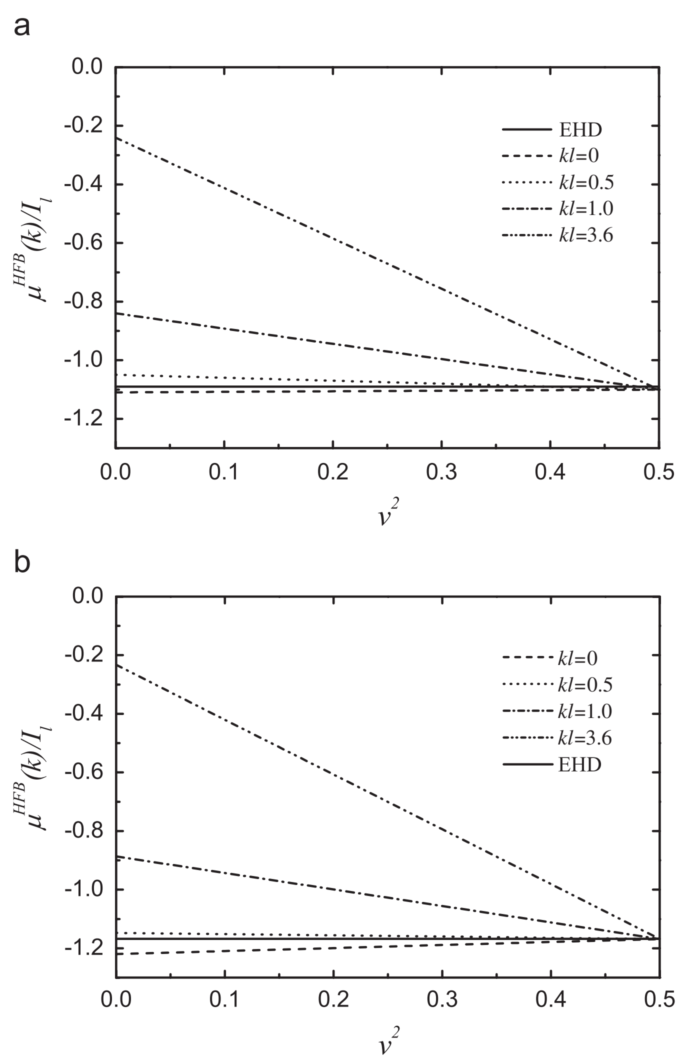

Fig. 3. Chemical potential versus filling factor v^2 for different values of the parameter $r = I_l/\hbar\omega_{\mathrm{c}}$. Solid line: energy per e–h pair in EHD phase; dashed line: chemical potential of condensed excitons with $k = 0$; dotted line: the same, but for $kl = 0.5$; dash-dotted line: the same, but for $kl = 1$; dash-dot–dot line: the same, but for $kl = 3.6$.

results illustrated in Fig. 1 with the ones obtained in this section. Putting the aim into practice we need to calculate the wave functions and the energy spectrum of the lowest exciton band. We will confine ourselves into the frame of four exciton levels model. By this restriction we will determine the influence of nearly situated exciton levels with the same wave vector on the lowest exciton level in which the BEC of magnetoexcitons takes place. These calculations are needed to obtain more exact expressions for the exciton wave function as well as of the exciton creation and annihilation operators. They will be represented as coherent superpositions of their zero order expressions and will lead us to the necessity to investigate the BEC in the exciton superposition state.

For the beginning we will define the magnetoexciton creation operator [25–27] characterized by the number n of the electron LL and by the number m of the hole LL, as well as by the 2D wave vector $\mathbf{k}$ with two components k_x and k_y

$$X^{\dagger}_{n,m,\mathbf{k}} = \frac{1}{\sqrt{N}}\sum_t \exp[-\mathrm{i}k_y t l^2] a^{\dagger}_{n,k_x/2+t} b^{\dagger}_{m,k_x/2-t}. \quad (43)$$

The state with the pair of numbers (n,m) equals to (0,0) will for simplicity be denoted as the state 1, the pair of numbers (1,1) gives rise to the magnetoexciton state 2, the pairs (1,0) and (0,1) will be mentioned as the states 3 and 4, correspondingly. The exciton wave function $\psi_i(k)$ is obtained as action of this operator on the vacuum state $|0\rangle$ determined as $a_{n,p}|0\rangle = 0,\ b_{m,p}|0\rangle = 0$

$$\psi_{i,\mathbf{k}} = X^{\dagger}_{n,m,\mathbf{k}}|0\rangle, \quad (44)$$

where i labels the quantum numbers $i \to (n,m)$. Thus, one can straightforwardly prove the orthogonality and normalization properties of eigenstates:

$$\langle\psi_{i,\mathbf{k}}|\psi_{j,\mathbf{k}'}\rangle = \delta_{i,j}\delta_{\mathbf{k},\mathbf{k}'}. \quad (45)$$

The matrix elements of the Hamiltonian Eq. (2), where the chemical potentials μ_{e} and μ_{h} are omitted as we are concerned in single exciton properties, on the wave functions Eq. (44) are denoted as

$$H_{ij}(k) = \langle\psi_{i,\mathbf{k}}|\hat{H}|\psi_{j,\mathbf{k}}\rangle = \delta_{i,j}\langle\psi_{i,\mathbf{k}}|\hat{H}_0|\psi_{i,\mathbf{k}}\rangle + V_{i,j}(\mathbf{k}), \quad (46)$$

where $V_{i,j}(\mathbf{k}) = \langle\psi_{i,\mathbf{k}}|\hat{H}_{\mathrm{Coul}}|\psi_{j,\mathbf{k}}\rangle$. Analytical expression for $V_{i,j}(\mathbf{k})$ one can derive from Eqs. (6) to (7) (for details see e.g. in Ref. [43]). The magnetoexciton energy bands will be determined at first in the zeroth order of the perturbation theory, when only the diagonal matrix elements H_{ii} and two off-diagonal matrix elements $V_{12}(k)$ and $V_{21}(k)$ are taken into account. The reason for the choice of the zero order approximation will be given below. All other off-diagonal matrix elements, which are smaller than the values V_{12} and V_{21} will be introduced in higher orders of the perturbation theory. We consider first the non-degenerate case when $\omega_{\mathrm{ce}} \neq \omega_{\mathrm{ch}}$,so that the four zero order magnetoexciton bands accounted from the LLLs are

$$E^1_{\mathrm{ex}}(k) = E^{(0,0)}_{\mathrm{ex}}(k) = H_{1,1}(k) = V_{1,1}(k) = -I^{(0,0)}_{\mathrm{ex}}(k), \quad (47a)$$

$$E^2_{\mathrm{ex}}(k) = E^{(1,1)}_{\mathrm{ex}}(k) = H_{2,2}(k) = \hbar\omega_{\mathrm{ce}} + \hbar\omega_{\mathrm{ch}} - I^{(1,1)}_{\mathrm{ex}}(k), \quad (47b)$$

$$E^3_{\mathrm{ex}}(k) = E^{(1,0)}_{\mathrm{ex}}(k) = H_{3,3}(k) = \hbar\omega_{\mathrm{ce}} - I^{(1,0)}_{\mathrm{ex}}(k), \quad (47c)$$

$$E^4_{\mathrm{ex}}(k) = E^{(0,1)}_{\mathrm{ex}}(k) = H_{4,4}(k) = \hbar\omega_{\mathrm{ch}} - I^{(0,1)}_{\mathrm{ex}}(k). \quad (47d)$$

The ionization potentials of four bands were determined as

$$I^{(n,m)}_{\mathrm{ex}}(k) = \sum_s F_{e-h}(p,n;k_x-p,m;p-s,n;k_x-p+s,m) \times \exp[\mathrm{i}k_y s l^2] = \frac{e^2}{\varepsilon_0 l}\int_0^{\infty}\mathrm{d}x \times \exp\left[-\frac{x^2}{2}\right]\left(1-\frac{x^2}{2}\right)^{n+m} \times J_0(klx), \quad n,m = 0,1. \quad (48)$$

An explicit expression for $I^{(n,m)}_{\mathrm{ex}}(k)$ can be found e.g. in Ref. [44].

A more exact expression of the magnetoexciton wave function is the linear combination

$$\psi_{\nu,\mathbf{k}} = \sum_{i=1}^{4} a_{i\nu}\psi_{i,\mathbf{k}}. \quad (49)$$

The new functions $\psi_{\nu,k}$ are denoted by Greek symbols $\nu = \alpha,\beta,\gamma,\delta$, whereas the initial zero order functions ψ_{ik} by Latin letter $i = 1,2,3,4$. They obey to the Schrödinger equation

$$\hat{H}\psi_{\nu,\mathbf{k}} = E_\nu(k)\psi_{\nu,\mathbf{k}} \quad (50)$$

what leads to four linear equations which determine the energy spectrum $E_\nu(k)$ and the coefficients $a_{i\nu}$. The equations contain the matrix elements Eq. (46) and have the form

$$\sum_{i=1}^{4} a_{i\nu}H_{ij}(k) = E_\nu(k)a_{j\nu}, \quad j = 1,2,3,4. \quad (51)$$

The energy spectrum $E_\nu(k)$ can be found solving secular equation

$$\begin{vmatrix} H_{11}(k)-E_\nu(k) & V_{12}(k) & V_{13}(k) & V_{14}(k) \\ V_{21}(k) & H_{22}(k)-E_\nu(k) & V_{23}(k) & V_{24}(k) \\ V_{31}(k) & V_{32}(k) & H_{33}(k)-E_\nu(k) & V_{34}(k) \\ V_{41}(k) & V_{42}(k) & V_{43}(k) & H_{44}(k)-E_\nu(k) \end{vmatrix} = 0. \quad (52)$$

Neglecting 10 off-diagonal matrix elements denoted as a first order infinitesimals ε

$$|V_{13}| \sim |V_{14}| \sim |V_{23}| \sim |V_{24}| \sim |V_{34}| \sim \varepsilon \quad (53)$$

we will obtain the zero order solutions of Eqs. (51) and (52)

$$E^0_{\alpha,\beta}(k) = \frac{H_{11}(k)+H_{22}(k)}{2} \mp \frac{1}{2}\sqrt{(H_{11}(k)-H_{22}(k))^2 + 4V^2_{12}(k)},$$
$$E^0_{\gamma}(k) = H_{33}(k), \quad E^0_{\delta}(k) = H_{44}(k). \quad (54)$$

In the limit $V_{12}(k) < \hbar\omega_{ce} + \hbar\omega_{ch}$ two magnetoexciton bands look as follows:

$$E^0_\alpha(k) = E^1_{ex}(k) - \frac{V^2_{12}(k)}{\hbar\omega_{ce} + \hbar\omega_{ch}},$$
$$E^0_\beta(k) = E^2_{ex}(k) + \frac{V^2_{12}(k)}{\hbar\omega_{ce} + \hbar\omega_{ch}}. \quad (55)$$

The coefficients $a^0_{i\nu}$ can be also determined in the zeroth order

$$|a^0_{1\alpha}|^2 = |a^0_{2\beta}|^2 = \frac{(H_{22}(k) - H_{11}(k))^2}{(H_{22}(k) - H_{11}(k))^2 + V^2_{12}(k)} \approx 1 - \frac{V^2_{12}(k)}{(\hbar\omega_{ce} + \hbar\omega_{ch})^2},$$
$$|a^0_{2\alpha}|^2 = |a^0_{1\beta}|^2 = \frac{V^2_{12}(k)}{(H_{22}(k) - H_{11}(k))^2 + V^2_{12}(k)} \approx \frac{V^2_{12}(k)}{(\hbar\omega_{ce} + \hbar\omega_{ch})^2},$$
$$a^0_{2\alpha} \approx -\frac{V_{12}(k)}{\hbar\omega_{ce} + \hbar\omega_{ch}}, a^0_{1\beta} \approx \frac{V_{12}(k)}{\hbar\omega_{ce} + \hbar\omega_{ch}},$$
$$a^0_{3\nu} = a^0_{4\nu} = 0 \quad \text{for } \nu = \alpha, \beta. \quad (56)$$

The different signs of the coefficients $a^0_{1\beta}$ and $a^0_{2\alpha}$ are taken to obey Eqs. (51) and will be important in the discussions below. We are interested in a more exact expressions for only the lowest exciton band taking into account the influence of three exciton bands situated upper on the energy scale. For these three bands involved in this scheme the starting wave function (49) is not sufficient because there are another exciton bands which do not affect too much the lowest exciton bands, but can stronger influence on the upper exciton bands. It can be easily shown that first order corrections give zero contribution to the energy and the following contribution to the coefficients $a_{i\nu}$:

$$a'_{1\nu} = a'_{2\nu} = 0,$$
$$a'_{3\nu} = -\frac{(V_{31}a^0_{1\nu} + V_{32}a^0_{2\nu})}{(H_{33} - E^0_\nu)}, \quad \nu = \alpha, \beta,$$
$$a'_{4\nu} = -\frac{(V_{41}a^0_{1\nu} + V_{42}a^0_{2\nu})}{(H_{44} - E^0_\nu)}. \quad (57)$$

The perturbation theory used here gives rise to the second order correction to the lowest exciton band E''_α

$$E''_\alpha(k) = \left[\frac{|V_{13}(k)|^2}{E^0_\alpha(k) - H_{33}(k)} + \frac{|V_{14}(k)|^2}{E^0_\alpha(k) - H_{44}(k)}\right]|a^0_{1\alpha}|^2 + \left[\frac{|V_{23}(k)|^2}{E^0_\alpha(k) - H_{33}(k)} + \frac{|V_{24}(k)|^2}{E^0_\alpha(k) - H_{44}(k)}\right]|a^0_{2\alpha}|^2 + a^{0*}_{1\alpha}a^0_{2\alpha}\left[\frac{V_{13}(k)V_{32}(k)}{E^0_\alpha(k) - H_{33}(k)} + \frac{V_{14}(k)V_{42}(k)}{E^0_\alpha(k) - H_{44}(k)}\right] + a^{0*}_{2\alpha}a^0_{1\alpha}\left[\frac{V_{23}(k)V_{31}(k)}{E^0_\alpha(k) - H_{33}(k)} + \frac{V_{24}(k)V_{41}(k)}{E^0_\alpha(k) - H_{44}(k)}\right]. \quad (58)$$

Substituting the coefficients $a^0_{1\alpha}$ and $a^0_{2\alpha}$, given by (56) and the matrix elements $V_{ij}(k)$ from (46) into (58) we obtain the dependence of $E''_\alpha(k)$ on the dimensionless wave vector kl. The dependence of the ionization potential of the lowest exciton band on the wave vector k with the account of the second order corrections is represented in Fig. 1 by the dotted and dash-dotted lines. This corrections are 40% smaller as compared with the influence of all ELL.

In the frame of four level models the magnetoexciton creation operator has the form

$$X^\dagger_{\alpha\mathbf{k}} = a^0_{1\alpha}X^\dagger_{0,0,\mathbf{k}} + a^0_{2\alpha}X^\dagger_{1,1,\mathbf{k}} + a'_{3\alpha}X^\dagger_{1,0,\mathbf{k}} + a'_{4\alpha}X^\dagger_{0,1,\mathbf{k}}. \quad (59)$$

On its base it is possible to construct a new displacement operator $\hat{D}(N_{ex})$ and to discuss the phenomenon of BEC taking into account explicitly the LLLs and FELLs.

Consider the BEC in the lowest exciton superposition state in a simplified variant, when only two main terms in expression (59) are taken into account and the coefficients of superposition are $a^0_{1\alpha} \equiv a_1$ and $a^0_{2\alpha} \equiv a_2$, with the condition $a_1^2 + a_2^2 = 1$. The expression Eq. (31) now can be rewritten as:

$$X^\dagger_{\mathbf{k}} = a_1 X^\dagger_{0,0,\mathbf{k}} + a_2 X^\dagger_{1,1,\mathbf{k}}. \quad (60)$$

The unitary transformation $\hat{D}(\sqrt{N_{ex}}) = \exp[\sqrt{N_{ex}}(X^\dagger(k) - X(k))]$ leads do generalized Bogoliubov's u-v transformation

$$\alpha_p = \hat{D}a_p\hat{D}^\dagger = \cos(ga_1)a_p - \sin(ga_1)\exp[-ik_y(p - k_x/2)]b^\dagger_{k_x-p},$$
$$\beta_p = \hat{D}a_p\hat{D}^\dagger = \cos(ga_1)b_p + \sin(ga_1)\exp[-ik_y(k_x/2 - p)]a^\dagger_{k_x-p},$$
$$\gamma_p = \hat{D}c_p\hat{D}^\dagger = \cos(ga_2)a_p - \sin(ga_2)\exp[-ik_y(p - k_x/2)]d^\dagger_{k_x-p},$$
$$\delta_p = \hat{D}d_p\hat{D}^\dagger = \cos(ga_2)d_p + \sin(ga_2)\exp[-ik_y(k_x/2 - p)]c^\dagger_{k_x-p}. \quad (61)$$

Here $g = \sqrt{2\pi l^2 n_{ex}}$; $n_{ex} = N_{ex}/S$; Fermi operators c_p, d_p and $c^\dagger_p$, $d^\dagger_p$ describe the annihilation and creation of an electron and a hole in the FELL, respectively. Then the new ground state wave function acquires the following form:

$$|\Psi_g(\mathbf{k})\rangle = \hat{D}(\sqrt{N_{ex}})|0\rangle = \prod_t(\cos(g) + \sin(g)\exp(-ik_y t l^2)[a_1 a^\dagger_{k_x/2+t}b^\dagger_{k_x/2-t} + a_2 c^\dagger_{k_x/2+t}d^\dagger_{k_x/2-t}]), \quad (62)$$

which plays the role of vacuum state for the operators $\alpha_p, \beta_p, \gamma_p, \delta_p$, i.e.

$$\alpha_p|\Psi_g(\mathbf{k})\rangle = \beta_p|\Psi_g(\mathbf{k})\rangle = \gamma_p|\Psi_g(\mathbf{k})\rangle = \delta_p|\Psi_g(\mathbf{k})\rangle = 0.$$

With the above definitions one can calculate the average number of excitons (which is equal to the average number of electrons or holes) as follows:

$$\langle \Psi_g(\mathbf{k})| \sum_t (a_t^\dagger a_t + c_t^\dagger c_t) |\Psi_g(\mathbf{k})\rangle = N[\sin^2(ga_1) + \sin^2(ga_2)]. \quad (63)$$

Owing to the definition of g one arrives at the concentration relation

$$g^2 = \sin^2(ga_1) + \sin^2(ga_2), \quad (64)$$

which in case of $a_2 = 0$ leads to the previous expression Eq. (36) of Ref. [27]. Another special case is the following:

$$a_1 = a_2 = 1/\sqrt{2}, \quad g^2/2 = \sin^2(g/\sqrt{2}), \quad (65)$$

what implies $g^2 < 2$. This relation generalize the description of the BEC of magnetoexciton gas on its superposition state, with electrons and holes residing both in the LLLs and the FELLs.

5. Conclusions

The virtual excitations due to Coulomb scattering of two charged particles from their LLLs to ELLs with arbitrary indices n and m and their return back to LLLs lead in the second order of the perturbation theory to supplementary indirect interaction between the particles side by side with their Coulomb interaction. General expressions for the corresponding matrix elements were obtained. On these base the influence of indirect interaction on the chemical potential of the condensed magnetoexcitons and on the energy per pair in the components of EHL and EHD were revealed in HFA. The indirect supplementary e–e and h–h interaction being averaged in HFA gives rise to direct pairing terms and exchanges pairing terms. The first terms being negative increase the binding energy of magnetoexcitons and energy per pair in the EHL phase, whereas the second terms are repulsive. They diminish the influence of the direct pairing terms, but do not surpass them, so that the resulting influence of both terms remains attractive. The supplementary e–h attraction after the u-v transformation in the case of BEC of magnetoexcitons in the state with wave vector $\mathbf{k}$ gives rise to repulsive-type Bogoliubov self-energy terms [41]. They stabilized the BEC in the small region of wave vectors $kl < 0.5$. Such terms do not appear in the case of EHL.

The energy per one e–h pair inside the EHD found to be situated on the energy scale very close to the value of the chemical potential of the Bose–Einstein condensed magnetoexcitons with wave vector $k = 0$ calculated in the HFBA. These two phases can coexist. Coexistence of the degenerate Bose gas with $k = 0$ and of the droplets of dielectric liquid phase formed by magnetoexcitons with non-zero wave vector $\mathbf{k}$ was revealed in Ref. [42], so that one can expect the coexistence of three phases simultaneously.

The wave functions of the lowest exciton levels in the second order of the perturbation theory represent the superpositions of the zero order exciton wave functions related with definite Landau levels. The BEC of magnetoexcitons in such superposition state involving different Landau levels was considered. The generalization of the Bogoliubov u-v transformations and of the BCS-type ground state wave function introduced by Keldysh and Kozlov in their electron–hole description of Bose–Einstein condensed excitons is proposed. On this base the phase-space filling factors of involved Landau levels were determined.

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