**THE EUROPEAN
PHYSICAL JOURNAL B**

Regular Article

Nambu-Goldstone modes of the two-dimensional Bose-Einstein condensed magnetoexcitons

S.A. Moskalenko[1], M.A. Liberman[2], D.W. Snoke[3], E.V. Dumanov[1,a], S.S. Rusu[1], and F. Cerbu[1]

[1] Institute of Applied Physics of the Academy of Sciences of Moldova, Academic Str. 5, 2028 Chisinau, Republic of Moldova
[2] Department of Physics, Uppsala University, Box 530, 75121 Uppsala, Sweden
[3] Department of Physics and Astronomy, University of Pittsburgh, 3941 O'Hara Street, 15260 Pittsburgh, USA

Abstract. The collective elementary excitations of two-dimensional magnetoexcitons in a Bose-Einstein condensate (BEC) with wave vector $\boldsymbol{k} = 0$ were investigated in the framework of the Bogoliubov theory of quasiaverages. The Hamiltonian of the electrons and holes lying in the lowest Landau levels (LLLs) contains supplementary interactions due to virtual quantum transitions of the particles to the excited Landau levels (ELLs) and back. As a result, the interaction between the magnetoexcitons with $\boldsymbol{k} = 0$ does not vanish and their BEC becomes stable. The equations of motion for the exciton operators $d(P)$ and $d^{\dagger}(P)$ are interconnected with equations of motion for the density operators $\rho(P)$ and $D(P)$. Instead of a set of two equations of motion, as in the case of usual Bose gas, corresponding to normal and abnormal Green's functions, we have a set of four equations of motion. This means we have to deal simultaneously with four branches of the energy spectrum, the two supplementary branches being the optical plasmon branch represented by the operator $\rho(P)$ and the acoustical plasmon branch represented by the operator $D(P)$. The perturbation theory on the small parameter $v^2(1 - v^2)$, where v^2 is the filling factor and $(1 - v^2)$ is the phase space filling factor was developed. The energy spectrum contains only one gapless, true Nambu-Goldstone (NG) mode of the second kind with dependence $\omega(k) \approx k^2$ at small values k describing the optical-plasmon-type oscillations. There are two exciton-type branches corresponding to normal and abnormal Green's functions. Both modes are gapped with roton-type segments at intermediary values of the wave vectors and can be named as quasi-NG modes. The fourth branch is the acoustical plasmon-type mode with absolute instability in the region of small and intermediary values of the wave vectors. All branches have a saturation-type dependencies at great values of the wave vectors. The number and the kind of the true NG modes is in accordance with the number of the broken symmetry operators. The comparison of the results concerning two Bose-Einstein condensates namely of the coplanar magnetoexcitons and of the quantum Hall excitons in the bilayer electron systems reveals their similarity.

1 Introduction

A two-dimensional electron system in a strong perpendicular magnetic field reveals fascinating phenomena such as the integer and fractional quantum Hall effects [1–5]. The discovery of the fractional quantum Hall effect (FQHE) fundamentally changed the established concepts about charged single-particle elementary excitations in solids [6–8]. The gauge transformations from the wave functions and creation operators of simple particles to another wave functions and creation operators describe composite particles (CPs) [9–11]. They are made from the previous particles and quantum vortices, created under the influence of the magnetic flux quanta [12–21]. Due to the contribution of many outstanding investigations [1–28] as well as many efforts to explain and to represent the

underlying processes in a more clear way [4,29,30] it is possible to make a short summary as follows. One can begin with the concept of composite particles proposed by Wilczek [9,10] in the form of particles with magnetic flux tubes attached. A posteriori, the flux tubes were substituted by quantum vortices, as was argued by Read in a series of papers [12–15]. In many explanations proposed by Enger [4], it was underlined that in 3D space the particles may obey only Fermi and Bose statistics, whereas in 2D space the fractional statistics are also possible. Now under the interchanging of two particles, the wave function obtains the phase factor $e^{i\pi\alpha}$ with any fractional values of α. Such particles were named "anyons" [9,10]. The gauge transformations [29] of the wave functions and of the creation and annihilation operators of the initial particles and the corresponding Hamiltonians was a powerful instrument revealing the fundamental physical processes hidden at the first sight in the quantum states of the system.

[a] e-mail: dum@phys.asm.md

The gauge transformation revealed the existence of the vortices in the system created by the magnetic flux quanta.

Girvin et al. [20,21] elaborated the theory of the collective elementary excitation spectrum in the case of the FQHE, closely analogous to Feynman's theory of superfluid helium. The predicted spectrum has a gap at $k = 0$ and a deep magneto-roton minimum at finite wavevector, which is a precursor to the gap collapse associated with Wigner crystal instability.

In this paper we study a coplanar electron-hole (e-h) system with electrons in a conduction band and holes in a valence band, both of which have Landau levels in a strong perpendicular magnetic field. Earlier, this system was studied in a series of papers mostly dedicated to the theory of 2D magnetoexcitons [31–43]. This system bears some resemblance to the case of a bilayer electron system [44–49]. A short review concerning the Bose-Einstein condensation (BEC) of the quantum Hall excitons (QHExs) arising in the bilayer electron systems in the conditions of the quantum Hall effect (QHE) at one half filling factor $\nu = 1/2$ for each layer and the total filling factor for two layer equal to unity $\nu_t = 1$ is needed. The aim of the review is to compare this phenomenon with the case of BEC of two-dimensional (2D) magnetoexcitons. Such comparison will permit to better understand the underlying physics and to verify the accuracy of the made approximations. In reference [44] Fertig investigated the energy spectrum of a bilayer electron systems in a strong perpendicular magnetic field and introduced the concept of the interlayer phase coherence of the electron states in two adjacent layers, which leads to the model of quantum Hall excitons in condition of their BEC. Unexpectedly a strong evidence of exciton BEC was ultimately found in a such surprising place as a double layer 2D electron system at high magnetic field [45]. In the QHE regime the excitons consist from electrons in the lowest Landau level (LLL) of the conduction band of one layer being bound to the holes which appear in the LLL of the conduction band in another layer. The ground state wave function proposed by Fertig [44] introduces the interlayer phase coherence reflecting a new state, in which the electrons are no-longer confined to one layer or to another, but instead of it they reside in coherent liniar combinations of the two layer states as follows

$$|\psi\rangle = \prod_t \left(u a^\dagger_{1t} + v a^\dagger_{2t} \right) |0\rangle, \quad u^2 + v^2 = 1. \quad (1)$$

The lowest levels of the Landau quantization in the Landau gauge are characterized by the quantum number $n = 0$ and the uni-dimensional wave number t. $|0\rangle$ is the vacuum state. The equality $u^2 = v^2 = 1/2$ reflects the half-filling of the LLL in each layer. Introducing the hole operator $d^\dagger_t$, d_t for the first layer instead of the operators $a^\dagger_{1t}$ and a_{1t} the function (1) was transcribed in the form

$$|\psi\rangle = \prod_t \left(u + v a^\dagger_t d^\dagger_{-t} \right) |\psi_0\rangle, \quad |\psi_0\rangle = \prod_t a^\dagger_{1t} |0\rangle$$
$$a_{2t} = a_t, \ a^\dagger_{2t} = a^\dagger_t, \ a_{1t} = d^\dagger_{-t}, \ a^\dagger_{1t} = d_{-t}. \quad (2)$$

The operators $a^\dagger_t d^\dagger_{-t}$ create the electron-hole pairs with total wave vector equal to zero. The wave function (2) can be interpreted as describing the BEC of the QHExs [44–49].

The system we are interested in has only one layer, with electrons in conduction band and holes in the valence band of the same layer created by optical excitation or by p-n doping injection (both of these methods can be called "pumping"). In this case there is an intrinsic metastability, since electrons in the conduction band can drop down into the valence band and recombine with holes there. But we assume that the recombination rate of the electrons with holes has such a slow rate that the number of electrons and holes is nearly conserved. Unlike the case of the bilayer electron system with a half-filled lowest Landau level, in the case of a single excited layer which we consider, the density of excitons can be quite low, so that the electron Landau level and the separate hole Landau level are each only slightly occupied, and Pauli exclusion and phase space filling do not come in to play.

Our result concerning the BEC at $T = 0$ are estimations able to describe the real situation at finite temperatures lower than the critical temperature of the Berezinskii-Kosterlitz-Thouless (BKT) topological phase transition [50–52] related with the existence of the vortices and their clusters such as bound vortex-antivortex pairs. Just the unbinding of these pairs determines the critical temperature $T_{BKT} = \frac{\pi n \hbar^2}{2mk_B}$, where n is the surface density of the Bose particles and m is their mass. On one side of the phase transition there is a quasi-ordered fluid and on the other is a disordered unbounded vortex plasma. Although the formation of an isolated vortex will not occur at low temperature due its extensive creation energy, there always can be production of a pair of vortices with equal and opposite charges since the perturbation produced by such a pair falls off sufficiently rapidly at large distances so that their energy is finite [51,52]. Such topological formations can be easily created by the thermal fluctuations. The presence of the vortex clusters makes the earlier infinite homogeneous 2D e-h system to be nonhomogeneous as a whole. But the local homogeneity with finite local surface areas can exist leading to the BEC with finite critical temperature $T_c = \frac{2\pi n \hbar^2}{mk_B \lg(nS)}$ (Ref. [53]). Instead of an off diagonal long-range order as in the case of 3D Bose-gas in the 2D systems, there is only a long rang correlations which decays algebraically with distance. In such a way the quantum vortices promote the BEC and the formation of the superfluid component of the 2D Bose-gas at finite temperatures and at the same time the superfluid component is necessary for the formation of the quantum vortices. It is a self-organization-type situation. The BKT phase transition is a widely studied phenomenon [54–56].

We are interested in the distribution of the flux quanta in the case of an electron-hole system with equal average numbers of electrons and holes $\bar{N}_e = \bar{N}_h$ with filling factor $\nu = \bar{N}_e/N$, where N is the total number of flux quanta. In the case of fractional integer filling factor there is an integer number of flux quanta per each e-h pair. The creation of the vortices in this case is not studied at the present time.
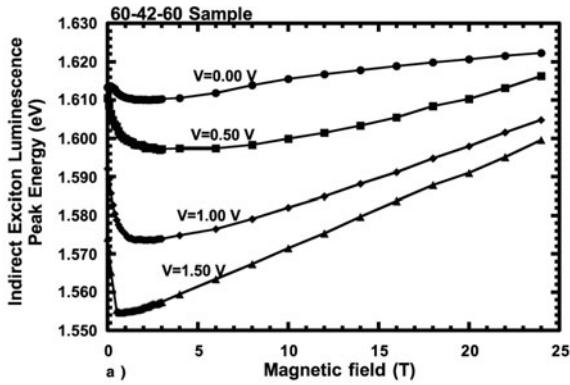
Fig. 1. Energy of the indirect exciton photon emission as a function of magnetic field, for several values of the electric field (from Ref. [58]).

Another related system is the case of a pumped bilayer system [57–67] such as coupled semiconductor quantum wells (CQW). While the geometry of this system with two adjacent layers is similar to the bilayer electron system discussed above, it is actually more similar in its main properties to the pumped single-layer electron-hole system which is the topic of this paper. In this system, the main effect of the bilayer structure is to put a tunneling barrier between the electrons which are in one layer and the holes which are in the other layer. This makes it possible to experimentally realize the condition given above of negligible electron-hole recombination; experimentally, exciton lifetimes of up to 40 microseconds have been observed [65,67], and the long lifetimes allow diffusion of the excitons over macroscopic distances [57] and equilibration to the lattice temperature in a harmonic potential [58]. Like the pumped single-layer system discussed here, the electrons in the conduction band and the holes in the valence band can each be at very low density. One drawback of this system is that it causes the interactions of the excitons to be enhanced, due to their alignment in the direction perpendicular to the layers, giving a strong dipole-dipole interaction [59]. For a comparison of the half-filled bilayer system and the pumped bilayer system, see reference [67].

In this bilayer system with nearly negligible recombination, a magnetic field can be applied in the direction perpendicular to the layers to create magnetoexcitons of the type considered here. So far, there has been no evidence of Bose-Einstein condensation in this system. The magnetoexciton regime can be reached, however, as shown in Figure 1. At high magnetic field, the exciton energy shifts up to higher energy following the Landau level energy, as expected. The fast shift to lower energy seen at low magnetic field has been explained as due to the effect of the disorder on the tunneling current in the samples [60].

Recently, another group has used magnetic field to study the Mott transition from insulating exciton gas to conducting plasma [61]. They were able to show that the magnetic response of the system changed sharply when the system had undergone a Mott transition. The Mott transition of exciton gas to plasma is in general still a quite difficult problem receiving much study [62,63].

One reason why BEC has not been observed clearly in either the simple pumped bilayer system or the pumped magnetoexciton bilayer system may be that the condensate occurs in a "dark" state which does not emit light. It has been proposed that condensation of excitons will always occur in a dark state if one exists [68–71]; some evidence of BEC of dark excitons has been reported in reference [65]. The existence of two types of excitons, dark and bright excitons, is related to their spin structure. In GaAs, the lowest exciton state splits into a $J = 2$ doublet which does not emit light, by symmetry, and a $J = 1$ doublet which emits light. In the present paper, we do not take into account the spin structure of the excitons.

Our paper is organized as follows. In Section 2 we deduce the Hamiltonian of the supplementary interactions between electrons and holes lying in the lowest Landau levels (LLLs) due to their virtual quantum transition from the LLLs to excited Landau levels (ELLs) in the processes of Coulomb scattering. It is interesting that this interaction due to the influence of the ELLs is possible to express in terms of two-particle operators in the form of density fluctuation operators. In Section 3 the full Hamiltonian containing the basic Coulomb interaction as well as the supplementary interaction is taken through the operation of gauge symmetry breaking so as to study the Bose-Einstein condensation (BEC) of magnetoexcitons. The traditional way is to follow the Keldysh-Kozlov-Kopaev method [72] describing the BEC of excitons in the electron-hole representation, which leads to the subsequent Bogoliubov u-v transformation of the creation and annihilation operators of the initial and final Fermi-type quasiparticles. In the present paper, we have chosen another variant proposed by Bogoliubov in his theory of quasiaverages [73]. Both methods are equivalent, but for our purposes the second variant is preferable because we will operate with the integral two-particle operators rather than with single-particle Fermi-type operators. In Section 4 we deduce the equations of motion for the integral two-particle operators and for the corresponding Green's functions. The truncation of the equations of motion permit us to obtain a Dyson equation in a 4×4 matrix form and to determine the self-energy parts Σ_{ij} with $i, j = 1, 2, 3, 4$. Section 5 is devoted to the analytical and numerical calculations of the energy spectrum in different approximations. The conclusions are presented in Section 6.

2 Hamiltonian of the supplementary interaction

The Hamiltonian of the Coulomb interaction of the electrons and holes lying on their LLLs has the form

$$\hat{H}_c = \frac{1}{2} \sum_{\boldsymbol{Q}} W_{\boldsymbol{Q}} \left[\hat{\rho}(\boldsymbol{Q})\hat{\rho}(-\boldsymbol{Q}) - \hat{N}_e - \hat{N}_h \right] - \mu_e \hat{N}_e - \mu_h \hat{N}_h.$$

(3)

Here $\hat{\rho}(\boldsymbol{Q})$ are the density fluctuation operators expressed through the electron $\hat{\rho}_e(\boldsymbol{Q})$ and hole $\hat{\rho}_h(\boldsymbol{Q})$ density

operators as follows

$$\hat{\rho}_e(\vec{Q}) = \sum_t e^{iQ_y t l^2} a^\dagger_{t-\frac{Q_x}{2}} a_{t+\frac{Q_x}{2}},$$

$$\hat{\rho}_h(\vec{Q}) = \sum_t e^{iQ_y t l^2} b^\dagger_{t+\frac{Q_x}{2}} b_{t-\frac{Q_x}{2}},$$

$$\hat{\rho}(\boldsymbol{Q}) = \hat{\rho}_e(\boldsymbol{Q}) - \hat{\rho}_h(-\boldsymbol{Q}),$$

$$\hat{D}(\boldsymbol{Q}) = \hat{\rho}_e(\boldsymbol{Q}) + \hat{\rho}_h(-\boldsymbol{Q}),$$

$$\hat{N}_e = \hat{\rho}_e(0), \quad \hat{N}_h = \hat{\rho}_h(0), \quad \hat{N} = \hat{N}_e + \hat{N}_h,$$

$$W_{\boldsymbol{Q}} = \frac{2\pi e^2}{\varepsilon_0 S |\boldsymbol{Q}|} e^{-Q^2 l^2/2}. \tag{4}$$

The density operators are integral two-particle operators. They are expressed through the single-particle creation and annihilation operators $a_p^\dagger, a_p$ for electrons and $b_p^\dagger, b_p$ for holes. ε_0 is the dielectric constant of the background; μ_e and μ_h are chemical potentials for electrons and holes.

The supplementary indirect interactions between electrons and holes appear due to the simultaneous virtual quantum transitions of two particles from the LLLs to excited Landau levels (ELLs) and their return back during the Coulomb scattering processes. This interaction was deduced in references [42,43]. It has a general attractive character and has the form

$$H_{\text{suppl}} = -\frac{1}{2} \sum_{p,q,s} \phi_{e-e}(p,q;s) a_p^\dagger a_q^\dagger a_{q+s} a_{p-s}$$

$$- \frac{1}{2} \sum_{p,q,s} \phi_{h-h}(p,q;s) b_p^\dagger b_q^\dagger b_{q+s} b_{p-s}$$

$$- \sum_{p,q,s} \phi_{e-h}(p,q;s) a_p^\dagger b_q^\dagger b_{q+s} a_{p-s}. \tag{5}$$

An important property of this quartic form constructed from single-particle operators is the possibility to transcribe it through the integral two-particle operators $\hat{\rho}(\boldsymbol{Q})$ and $\hat{D}(\boldsymbol{Q})$ as follows (Ref. [74])

$$H_{\text{suppl}} = \frac{1}{2} B_{i-i} \hat{N} - \frac{1}{4N} \sum_Q V(Q) \hat{\rho}(\boldsymbol{Q}) \hat{\rho}(-\boldsymbol{Q})$$

$$- \frac{1}{4N} \sum_Q U(Q) \hat{D}(\boldsymbol{Q}) \hat{D}(-\boldsymbol{Q}). \tag{6}$$

The estimates give the values (Ref. [42,43,74])

$$U(\boldsymbol{Q}) \cong U(0) e^{-Q^2 l^2/2}, \quad U(0) = 2A_{i-i},$$

$$\frac{1}{N} \sum_{\boldsymbol{Q}} U(\boldsymbol{Q}) = B_{i-i} + \Delta(0),$$

$$V(\boldsymbol{Q}) \approx V(0) = 0, \quad A_{i-i} = 0.481 \frac{I_l^2}{\pi \hbar \omega_c},$$

$$B_{i-i} = 0.432 \frac{I_l^2}{\pi \hbar \omega_c}, \quad \Delta(0) = 0.688 \frac{I_l^2}{\pi \hbar \omega_c}. \tag{7}$$

Here I_l is the ionization potential of magnetoexciton, $\hbar \omega_c$ is the cyclotron frequency at $m_e = m_h$.

The full Hamiltonian describing the interaction of electrons and holes lying on the LLLs is

$$H = H_{Coul} + H_{\text{suppl}}. \tag{8}$$

3 Breaking of the gauge symmetry. Bose-Einstein condensation of magnetoexcitons

In references [32–41] the Bose-Einstein condensation (BEC) of magnetoexcitons with wave vector $\boldsymbol{k}$ different from zero was considered without taking into account the influence of the excited Landau levels (ELLs). But the case of BEC with $\boldsymbol{k} = 0$ was impossible to incorporate in the previous description because such magnetoexcitons form an ideal Bose-gas: in that model, the interaction between two magnetoexcitons with electrons and holes lying on the LLLs and with $\boldsymbol{k} = 0$ equals exactly zero. This prediction of ideal behavior is a result of the made assumptions and approximations neglecting the influence of the excited Landau levels. To better understand the properties of the 2D magnetoexcitons with $\boldsymbol{k} = 0$ one can imagine the electron and hole of the one exciton counter propagating for example in the x in-plane direction in the perpendicular magnetic field. They are shifted by the Lorentz force in the same part of the in-plane y direction and undergo cyclotron quantization around the same gyration point. Their radii of the cyclotron motion are exactly the same because they are determined by the magnetic length l and not by the masses of the particles. The 2D magnetoexciton with $\boldsymbol{k} = 0$ represents two cyclotron orbits of the electron and of the hole exactly overposed in the same gyration point. The Coulomb interaction of the e-h pair of the one such exciton with the e-h pair of another such exciton exactly equals to zero. To consider BEC of magnetoexcitons with $\boldsymbol{k} = 0$ it is necessary to take into account the influence of the ELLs, represented by the Hamiltonian in equation (6).

As discussed in previous papers [33–43,75], the breaking of the gauge symmetry of the Hamiltonian (8) can be achieved using the Keldysh-Kozlov-Kopaev method [72] using the unitary transformation

$$\hat{D}(\sqrt{N_{ex}}) = \exp[\sqrt{N_{ex}}(d^\dagger(\boldsymbol{k}) - d(\boldsymbol{k}))], \tag{9}$$

where $d^\dagger(k)$ and $d(k)$ are the creation and annihilation operators of the magnetoexcitons. In the electron-hole representation they are (Refs. [33–43]):

$$d^\dagger(\boldsymbol{P}) = \frac{1}{\sqrt{N}} \sum_t e^{-iP_y t l^2} a^\dagger_{t+\frac{P_x}{2}} b^\dagger_{-t+\frac{P_x}{2}},$$

$$d(\boldsymbol{P}) = \frac{1}{\sqrt{N}} \sum_t e^{iP_y t l^2} b_{-t+\frac{P_x}{2}} a_{t+\frac{P_x}{2}}, \tag{10}$$

BEC of magnetoexcitons leads to the formation of a coherent macroscopic state as a ground state of the system with wave function

$$|\psi_g(\bar{k})\rangle = \hat{D}(\sqrt{N_{ex}}) |0\rangle, \quad a_p |0\rangle = b_p |0\rangle = 0. \tag{11}$$

Here $|0\rangle$ is the vacuum state for electrons and holes. In spite of the fact that we kept arbitrary value of $\boldsymbol{k}$, nevertheless our main goal is the BEC with $\boldsymbol{k} = 0$ and we will consider the interval $0.5 > kl \geq 0$. The function (11) will be used in Section 5 to calculate the averages values of the type $\left\langle D(\overrightarrow{Q})D(-\overrightarrow{Q})\right\rangle$. The transformed Hamiltonian (8) looks like

$$\hat{\mathcal{H}} = D\left(\sqrt{N_{ex}}\right) H D^{\dagger}\left(\sqrt{N_{ex}}\right), \qquad (12)$$

and is succeeded, as usual, by the Bogoliubov u-v transformations of the single-particle Fermi operators

$$\alpha_p = \hat{D}\left(\sqrt{N_{ex}}\right) a_p \hat{D}^{\dagger}\left(\sqrt{N_{ex}}\right) = u a_p - v(p - \frac{k_x}{2}) b_{k_x - p}^{\dagger},$$
$$\alpha_p \left|\psi_g(\bar{k})\right\rangle = 0,$$
$$\beta_p = \hat{D}\left(\sqrt{N_{ex}}\right) b_p \hat{D}^{\dagger}\left(\sqrt{N_{ex}}\right) = u b_p + v(\frac{k_x}{2} - p) a_{k_x - p}^{\dagger},$$
$$\beta_p \left|\psi_g(\bar{k})\right\rangle = 0. \qquad (13)$$

Instead of this traditional way of transforming the expressions of the starting Hamiltonian (8) and of the integral two-particle operators (4) and (10), we will use the method proposed by Bogoliubov in his theory of quasi-averages [73], remaining in the framework of the original operators. The new variant is completely equivalent to the previous one, and both of them can be used in different stages of the calculations. For example, the average values can be calculated using the wave function (11) and u-v transformations (13), whereas the equations of motion for the integral two-particle operators can be simply written in the starting representation.

The Hamiltonian (8) with the broken gauge symmetry in the lowest approximation has the form

$$\hat{\mathcal{H}} = \frac{1}{2}\sum_{\boldsymbol{Q}} W_{\boldsymbol{Q}}\left[\rho(\boldsymbol{Q})\rho(-\boldsymbol{Q}) - \hat{N}_e - \hat{N}_h\right] - \mu_e \hat{N}_e$$
$$- \mu_h \hat{N}_h + \frac{1}{2}B_{i-i}\hat{N} - \frac{1}{4N}\sum_Q V(Q)\hat{\rho}(\boldsymbol{Q})\hat{\rho}(-\boldsymbol{Q})$$
$$- \frac{1}{4N}\sum_Q U(Q)\hat{D}(\boldsymbol{Q})\hat{D}(-\boldsymbol{Q}) - \tilde{\eta}\sqrt{N}\left(d^{\dagger}(k) + d(k)\right). \qquad (14)$$

Another smaller term, proportional to $\tilde{\eta}$, has been dropped for simplicity.

Here the parameter $\tilde{\eta}$, which determines the breaking of the gauge symmetry, depends, as in the case of weakly non-ideal Bose-gas considered by Bogoliubov [73], on the chemical potential μ and on the square root of the density. In our case the density is proportional to the filling factor $\nu = v^2$. We have:

$$\mu = \mu_e + \mu_h, \quad \bar{\mu} = \mu + I_l, \quad N_{ex} = v^2 N,$$
$$\tilde{E}_{ex}(k) = -I_l - \Delta(k) + E(k),$$
$$\tilde{\eta} = (\tilde{E}_{ex}(k) - \mu)v = (E(k) - \Delta(k) - \bar{\mu})v,$$
$$E(k) = 2\sum_Q W_Q \sin^2\left(\frac{[K \times Q]_z l^2}{2}\right). \qquad (15)$$

In the special case $\boldsymbol{k} = 0$ we obtain

$$\tilde{\eta} = -(\bar{\mu} + \Delta(0))v. \qquad (16)$$

4 Equations of motion for the integral two-particle operators. Green's functions, Dyson equation and Self-energy parts in the case k = 0

The equations of motion for the integral two-particle operators with wave vectors $\boldsymbol{P} \neq 0$ in the special case of BEC of magnetoexcitons with $\boldsymbol{k} = 0$ are

$$i\hbar\frac{d}{dt}d(\boldsymbol{P}) = [d(\boldsymbol{P}), \hat{\mathcal{H}}]$$
$$= (-\bar{\mu} + E(\boldsymbol{P}) - \Delta(\boldsymbol{P}))d(\boldsymbol{P})$$
$$- 2i\sum_{\boldsymbol{Q}} \tilde{W}(\boldsymbol{Q})\sin\left(\frac{[\boldsymbol{P} \times \boldsymbol{Q}]_z l^2}{2}\right)$$
$$\times \hat{\rho}(\boldsymbol{Q})d(\boldsymbol{P} - \boldsymbol{Q}) - \frac{1}{N}$$
$$\times \sum_{\boldsymbol{Q}} U(\boldsymbol{Q})\cos\left(\frac{[\boldsymbol{P} \times \boldsymbol{Q}]_z l^2}{2}\right)$$
$$\times D(\boldsymbol{Q})d(\boldsymbol{P} - \boldsymbol{Q}) + \tilde{\eta}\frac{D(\boldsymbol{P})}{\sqrt{N}},$$

$$i\hbar\frac{d}{dt}d^{\dagger}(-\boldsymbol{P}) = [d^{\dagger}(-\boldsymbol{P}), \hat{\mathcal{H}}]$$
$$= (\bar{\mu} - E(-\boldsymbol{P}) + \Delta(-\boldsymbol{P}))d^{\dagger}(-\boldsymbol{P})$$
$$+ 2i\sum_{\boldsymbol{Q}} \tilde{W}(\boldsymbol{Q})\sin\left(\frac{[\boldsymbol{P} \times \boldsymbol{Q}]_z l^2}{2}\right)$$
$$\times d^{\dagger}(-\boldsymbol{P} - \boldsymbol{Q})\hat{\rho}(-\boldsymbol{Q})$$
$$+ \frac{1}{N}\sum_{\boldsymbol{Q}} U(\boldsymbol{Q})\cos\left(\frac{[\boldsymbol{P} \times \boldsymbol{Q}]_z l^2}{2}\right)$$
$$\times d^{\dagger}(-\boldsymbol{P} - \boldsymbol{Q})D(-\boldsymbol{Q}) - \tilde{\eta}\frac{D(\boldsymbol{P})}{\sqrt{N}}, \qquad (17)$$

$$i\hbar\frac{d}{dt}\hat{\rho}(\boldsymbol{P}) = [\hat{\rho}(\boldsymbol{P}), \hat{\mathcal{H}}]$$
$$= -i\sum_{\boldsymbol{Q}} \tilde{W}(\boldsymbol{Q})\sin\left(\frac{[\boldsymbol{P} \times \boldsymbol{Q}]_z l^2}{2}\right)$$
$$\times [\hat{\rho}(\boldsymbol{P} - \boldsymbol{Q})\hat{\rho}(\boldsymbol{Q}) + \hat{\rho}(\boldsymbol{Q})\hat{\rho}(\boldsymbol{P} - \boldsymbol{Q})]$$
$$+ \frac{i}{2N}\sum_{\boldsymbol{Q}} U(\boldsymbol{Q})\sin\left(\frac{[\boldsymbol{P} \times \boldsymbol{Q}]_z l^2}{2}\right)$$
$$\times [D(\boldsymbol{P} - \boldsymbol{Q})D(\boldsymbol{Q}) + D(\boldsymbol{Q})D(\boldsymbol{P} - \boldsymbol{Q})],$$

$$i\hbar\frac{d}{dt}\hat{D}(\boldsymbol{P}) = [\hat{D}(\boldsymbol{P}), \hat{\mathcal{H}}]$$

$$= -i\sum_{\boldsymbol{Q}} \tilde{W}(\boldsymbol{Q}) \sin\left(\frac{[\boldsymbol{P}\times\boldsymbol{Q}]_z l^2}{2}\right)$$

$$\times [\hat{\rho}(\boldsymbol{Q})\hat{D}(\boldsymbol{P}-\boldsymbol{Q}) + \hat{D}(\boldsymbol{P}-\boldsymbol{Q})\hat{\rho}(\boldsymbol{Q})]$$

$$+ \frac{i}{2N}\sum_{\boldsymbol{Q}} U(\boldsymbol{Q}) \sin\left(\frac{[\boldsymbol{P}\times\boldsymbol{Q}]_z l^2}{2}\right)$$

$$\times [\hat{D}(\boldsymbol{Q})\hat{\rho}(\boldsymbol{P}-\boldsymbol{Q}) + \hat{\rho}(\boldsymbol{P}-\boldsymbol{Q})\hat{D}(\boldsymbol{Q})]$$

$$+ 2\tilde{\eta}\sqrt{N}\left[d(\boldsymbol{P}) - d^{\dagger}(-\boldsymbol{P})\right].$$

Following the equations of motion (17) we have introduce four interconnected retarded Green's functions at $T = 0$ [76,77]

$$G_{11}(\boldsymbol{P},t) = \left\langle\left\langle d(\boldsymbol{P},t); \hat{X}^{\dagger}(\boldsymbol{P},0)\right\rangle\right\rangle,$$

$$G_{12}(\boldsymbol{P},t) = \left\langle\left\langle d^{\dagger}(-\boldsymbol{P},t); \hat{X}^{\dagger}(\boldsymbol{P},0)\right\rangle\right\rangle,$$

$$G_{13}(\boldsymbol{P},t) = \left\langle\left\langle \frac{\hat{\rho}(\boldsymbol{P},t)}{\sqrt{N}}; \hat{X}^{\dagger}(\boldsymbol{P},0)\right\rangle\right\rangle,$$

$$G_{14}(\boldsymbol{P},t) = \left\langle\left\langle \frac{\hat{D}(\boldsymbol{P},t)}{\sqrt{N}}; \hat{X}^{\dagger}(\boldsymbol{P},0)\right\rangle\right\rangle, \quad (18)$$

as well as their Fourier transforms $G_{ij}(\boldsymbol{P},\omega)$, for which the equations of motion of the same type as the equations of motion (17) were obtained. These Green's functions can be called one-operator Green's functions, because they contain only one two-particle operator of the type $d^{\dagger}$, d, ρ, D. But on the right hand side of the corresponding equations of motion there is a second generation of two-operator Green's functions, containing the different products of the two-particle operators mentioned above. For them, the second generation of the equations of motion was deduced, containing in their right hand sides the Green's function of the third generation. They are three-operator Green's functions for which it is necessary to deduce the third generation of equations of motion. But we have stopped here the evolution of the infinite chains of equations of motion for multi-operators Green's function following the procedure proposed by Zubarev [77]. The truncation of the chains of the equations of motion and the decoupling of the one-operator Green's functions from the multi-operator Green's functions was achieved by substituting the three-operator Green's functions by one-operator Green's functions multiplied by the average value of the remaining two operators. The average values were calculated using the ground state wave function (11) and the u-v transformations (13). The Zubarev procedure is equivalent to a perturbation theory with a small parameter of the type $v^2(1-v^2)$, which represent the product of a filling factor $\nu = v^2$ and the phase-space filling factor $(1-v^2)$ reflecting the Pauli exclusion principle.

The close system of Dyson equations has the form

$$\sum_{j=1}^{4} G_{1j}(\boldsymbol{P},\omega)\Sigma_{jk}(\boldsymbol{P},\omega) = C_{1k}, \quad k = 1, 2, 3, 4. \quad (19)$$

There are 16 different components of the self-energy parts $\Sigma_{jk}(\boldsymbol{P},\omega)$ forming a 4×4 matrix.

5 Energy spectrum of collective elementary excitations

The self-energy parts entering into the Dyson equation (19) contain the average values of two-operator products, which appeared after the decoupling of the three-operator Green's functions by expressing them through one-operator Green's functions. The average values were calculated using the ground state wave function (11) and the coefficients of the u-v transformation (13). For operators $\hat{D}(\boldsymbol{Q})$, $\hat{\rho}(\boldsymbol{Q})$, $d(\boldsymbol{Q})$ and $d^{\dagger}(\boldsymbol{Q})$ with $\boldsymbol{Q} \neq 0$ we have

$$\langle D(\boldsymbol{Q})D(-\boldsymbol{Q})\rangle = 4u^2v^2N,$$

$$\langle \rho(\boldsymbol{Q})\rho(-\boldsymbol{Q})\rangle = 0,$$

$$\langle D(\boldsymbol{Q})d(-\boldsymbol{Q})\rangle = \langle d^{\dagger}(\boldsymbol{Q})D(-\boldsymbol{Q})\rangle = -2uv\sqrt{N},$$

$$\langle \rho(\boldsymbol{Q})d(-\boldsymbol{Q})\rangle = \langle d^{\dagger}(\boldsymbol{Q})\rho(-\boldsymbol{Q})\rangle = 0,$$

$$\langle d(0)\rangle = \langle d^{\dagger}(0)\rangle = uv\sqrt{N},$$

$$u^2 + v^2 = 1, \quad N = \frac{S}{2\pi l^2}. \quad (20)$$

The averages (20) have extensive values depending on N. For the condensate with $\boldsymbol{k} = 0$ they do not depend on the wave vector $\boldsymbol{Q}$. In the point $\boldsymbol{Q} = 0$ their values do not coincide with (20), changing by a jump. The expressions (20) have a small parameter of the perturbation theory in the forms $u^2v^2 = v^2(1-v^2)$, uv^3 and uv. The chemical potential $\bar{\mu}$ and the parameter $\tilde{\eta}$ of the quasiaverage theory are

$$\bar{\mu} + \Delta(0) = 2v^2(B_{i-i} - 2A_{i-i} + \Delta(0)) = 0.1rI_lv^2,$$

$$\tilde{\eta} = -0.1rI_lv^3,$$

$$0 < r = \frac{I_l}{\hbar\omega_c} \leq 1, \quad I_l = \frac{2\pi e^2}{\varepsilon_0 l}\sqrt{\frac{\pi}{2}}. \quad (21)$$

Here l is the magnetic length, ε_0 is the dielectric constant of the background and I_l is the ionization potential of the magnetoexciton. The chemical potential $\bar{\mu}$ has an increasing dependence on the concentration of electrons n_e or of magnetoexcitons n_{ex}, which are proportional to the filling factor $\nu = v^2$ as follows

$$n_e = n_{ex} = \frac{v^2}{2\pi l^2}. \quad (22)$$

It means that the system of Bose-Einstein condensed magnetoexcitons with wave vector $\boldsymbol{k} = 0$ is stable against collapse. Its stability is completely due to the influence of

the ELLs. In spite of the fact that the supplementary interaction has an overall attractive character and its average values in the Hartree approximation remain attractive, nevertheless the exchange-type Fock terms as well as other terms arising due to the Bogoliubov u-v transformation give rise to a repulsive interaction in the system, which is necessary to stabilize the BEC of magnetoexcitons with $\boldsymbol{k} = 0$. The role of the exchange Fock terms supplying a repulsion in conditions of overall attraction is similar to the case of the electron plasma, when the exchange Fock terms supply an effective attraction and lowering of the energy per particle in the condition of an overall Coulomb repulsion between the electrons [78]. The influence of the exchange Fock terms is the same in the case of electron-hole liquid (EHL) [79]. Another system with mixed interactions, the Wannier-Mott excitons with different spin projections, was investigated in reference [80]. It was shown that the coherent pairing of excitons leads to Bose-Einstein condensation of biexcitons. There the excitons with opposite spin projections and attractive interaction formed the biexcitons, whereas other excitons with parallel spin projections and repulsive interactions stabilized the Bose-Einstein condensate of biexcitons. The situation with magnetoexcitons also does not coincide exactly with the Bogoliubov model of a weakly non-ideal Bose-gas with pure repulsive interaction, because the presence of the attractive Hartree terms mentioned above.

The influence of the ELLs and their stabilizing role in the theory of the BEC of magnetoexcitons with nonzero wave vector $\boldsymbol{k}$ decreases quickly with the increasing of kl. In the range $kl \geq 0.5$ the Bose-Einstein condensate becomes unstable in the Hartree-Fock-Bogoliubov approximation [42,43]. Only in the range of $kl \sim 3-4$ the ability to stabilize the condensate does appear, taking into account the Anderson-type coherent excited states and the correlation energy calculated on this base. Under these conditions a metastable dielectric liquid phase (MDLP) formed by Bose-Einstein condensed magnetoexcitons with $kl \sim 3-4$ was found [39]. The collective elementary excitations under these conditions were investigated in reference [75].

In the case $k = 0$ the self-energy parts entering into the Dyson equation (19) contain only the coefficients linear in $U(P)$, or quadratic dependencies of the types $U(Q)U(-Q)$ and $\tilde{W}(Q)U(-Q-P)$ which reflects the influence of ELLs.

The equations of motion for the exciton operators $d(P)$ and $d^{\dagger}(-P)$ are interconnected with equations of motion for the density operators $\hat{\rho}(P)$ and $\hat{D}(P)$. Instead of a set of two equations of motion as in the case of the usual Bose-gas corresponding to normal and abnormal Green's functions, we have a set of four equations of motion. Changing the center-of-mass wave vector of the magnetoexciton, for example, from 0 to $\boldsymbol{P}$, means changing its internal structure, because the internal distance between the Landau orbits of the quantized electron and hole becomes equal to $|\boldsymbol{P}|l^2$. The separated electrons and holes remaining in their Landau orbits can take part in the formation of magnetoexcitons as well as in collective plasma oscillations. Such possibilities were not considered in the theory of structureless bosons or in the case of Wannier-Mott excitons with a rigid relative electron-hole motion structure without the possibility of the intra-series excitations. In the case of magnetoexcitons, their internal structure is much less rigid than in the case of Wannier-Mott excitons and the possibilities for electrons and holes to take part simultaneously in many processes are much more diverse. Instead of the branches of the energy spectrum corresponding to normal and abnormal Green's functions we have dealt simultaneously with four branches of the energy spectrum, the two supplementary branches being the optical plasmon branch represented by the operator $\hat{\rho}(P)$ and the acoustical plasmon branch represented by the operator $\hat{D}(P)$. One can see that the equations of motion for the operators $d(P)$ and $d^{\dagger}(-P)$ reflect the interaction of excitons with optical and acoustical plasmons but do not contain the direct interaction between themselves. The interaction with acoustical plasmons also takes place through the quasiaverage constant $\tilde{\eta}$.

The equation of motion for the acoustical plasmon operator $\hat{D}(P)$ contains the interaction with optical plasmons and the direct interaction with the magnetoexcitons. The optical plasmon motion is more separated from the equations of motion of other partners. It does not contain the direct interconnection with exciton branches, and the dispersion equation for the optical plasmon will be separated from the dispersion equation of the other three partners. In spite of this, optical plasmon branches are also influenced by the ground state of the system formed by the Bose-Einstein condensed magnetoexcitons with $k = 0$, because the self-energy part $\Sigma_{33}(\boldsymbol{P}, \omega)$ also depends on the averages (20). On the contrary, the equation of motion of the acoustical plasmon operator is closely interconnected with the equation of motion of the both exciton operators and its energy spectrum cannot be found out separately. Unlike the usual theory of a Bose-gas, the interaction between the magnetoexciton branches is not direct, but indirect, being mediated by the direct interaction with the plasmons. These peculiarities make our case different from those considered earlier in references [72,73,81].

The cumbersome dispersion equation is expressed in general form by the determinant equation

$$\det |\Sigma_{ij}(\boldsymbol{P}, \omega)| = 0, \qquad (23)$$

due to the structure of the self-energy parts. It separates into two independent equations. One of them concerns only the optical plasmon branch and has a simple form

$$\Sigma_{33}(\boldsymbol{P}, \omega) = 0. \qquad (24)$$

It does not include at all the chemical potential $\bar{\mu}$ and the quasiaverage constant $\tilde{\eta}$. The second equation contains the self-energy parts Σ_{11}, Σ_{22}, Σ_{44}, Σ_{14}, Σ_{41}, Σ_{24} and Σ_{42}, which include the both parameters $\bar{\mu}$ and $\tilde{\eta}$. The second equation has the form

$$\begin{aligned}
&\Sigma_{11}(\boldsymbol{P}; \omega)\Sigma_{22}(\boldsymbol{P}; \omega)\Sigma_{44}(\boldsymbol{P}; \omega) \\
&- \Sigma_{41}(\boldsymbol{P}; \omega)\Sigma_{22}(\boldsymbol{P}; \omega)\Sigma_{14}(\boldsymbol{P}; \omega) \\
&- \Sigma_{42}(\boldsymbol{P}; \omega)\Sigma_{11}(\boldsymbol{P}; \omega)\Sigma_{24}(\boldsymbol{P}; \omega) = 0.
\end{aligned} \qquad (25)$$

The solution of the equation (24) is

$$
\begin{aligned}
(\hbar\omega(P))^2 = {} & \frac{\langle D(P)D(-P)\rangle}{N^2} \\
& \times \sum_Q U(Q)\left(U(-Q) - U(Q-P)\right) \\
& \times \sin^2\left(\frac{[P\times Q]_z l^2}{2}\right).
\end{aligned}
\tag{26}
$$

The right hand side of this expression at small values of P has a dependence $|P|^4$ and tends to saturate at large values of P. The optical plasmon branch $\hbar\omega_{OP}(P)$ has a quadratic dispersion law in the long wavelength limit and saturation dependence in the range of short wavelengths. Its concentration dependence is of the type $\sqrt{v^2(1-v^2)}$ what coincides with the concentration dependencies for 3D plasma $\omega_p^2 = \dfrac{4\pi e^2 n_e}{\varepsilon_0 m}$ (Ref. [78]) and for 2D plasma $\omega_p^2(q) = \dfrac{2\pi e^2 n_s q}{\varepsilon_0 m}$ (Ref. [82]), where n_e and n_s are the corresponding electron densities. The supplementary factor $(1-v^2)$ in our case reflects the Pauli exclusion principle and the vanishing of the plasma oscillations at $\nu = v^2 = 1$. The obtained dispersion law is represented in Figure 2. The similar dispersion law was obtained in the case of 2D electron-hole liquid (EHL) in a strong perpendicular magnetic field [83], when the influence of the quantum vortices created by electron and hole subsystems is compensated exactly. But the saturation dependencies in these two cases are completely different. In the case of Bose-Einstein condensed magnetoexcitons it is determined by the ELLs, whereas in the case of EHL [83] it is determined by the Coulomb interaction in the frame of the LLLs.

The solutions of the dispersion equation (25) describe the exciton energy branch, the exciton quasienergy branch and the acoustical plasmon branch. The ideal magnetoexciton gas can exist only in the case $v^2 = 0$, with an infinitesimal number of excitons, but without plasma at all. The real parts $\sigma_{ij}(\boldsymbol{P},\omega)$ of the self-energy parts $\Sigma_{ij}(\boldsymbol{P},\omega)$ are

$$
\begin{aligned}
&\sigma_{11}(\boldsymbol{P},\omega) = \hbar\omega + \bar\mu - E(P) + \Delta(P), \quad \bar\mu + \Delta(0) = 0, \\
&\sigma_{22}(\boldsymbol{P},\omega) = \hbar\omega - \bar\mu + E(P) - \Delta(-P), \quad \tilde\eta = 0, \\
&\sigma_{33}(\boldsymbol{P},\omega) = \sigma_{44}(\boldsymbol{P},\omega) = \hbar\omega, \quad \Delta(P) \approx \Delta(0).
\end{aligned}
\tag{27}
$$

The excitation of magnetoexcitons means to transfer one of them from the ground state with energy $-I_l$ to the excited state $-I_l + E(P)$. For this reason the magnetoexciton excitation spectrum equals $\hbar\omega_{ex}(P) = \pm E(P)$, whereas the plasma oscillation frequency vanishes $\hbar\omega = 0$. This ideal variant is represented in Figure 3. In the case of a non-ideal Bose-gas with $\nu \neq 0$ the self-energy parts contain terms linear in $U(P)$ and a term quadratic in the interaction constant with unknown frequency in the denominators under the summation symbols. Such terms increase the number of the solutions, but can also be taken into account by an iteration method. In this case one can obtain corrections to the earlier solutions.
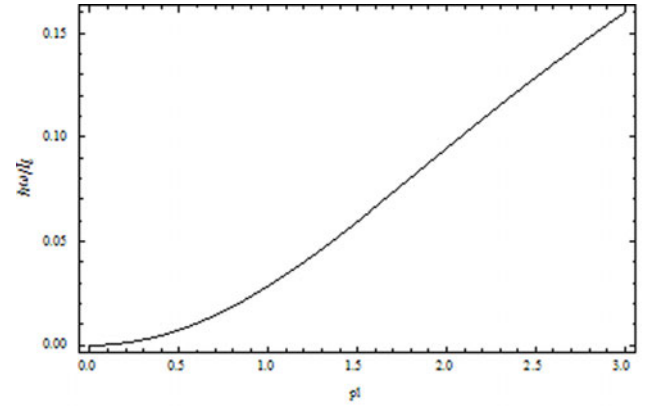

Fig. 2. The energy spectrum of the optical plasmon branch in the system of Bose-Einstein condensed (BEC-ed) magnetoexcitons with wave vector $\boldsymbol{k} = 0$, filling factor $\nu = v^2 = 0.1$, under the influence of excited Landau levels with parameter $r = 1/2$.
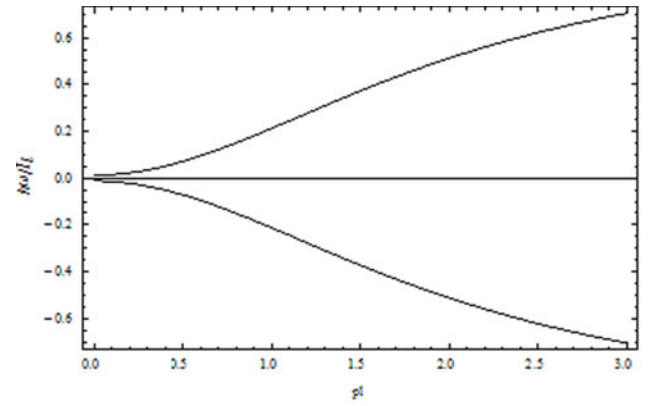

Fig. 3. The energy spectrum of the exciton branches in the case of an ideal BEC-ed magnetoexcitons gas with $\boldsymbol{k} = 0$ and filling factor equal to zero.

The first step in this procedure gives the real parts of the self-energy parts

$$
\begin{aligned}
&\sigma_{11}(\boldsymbol{P},\omega) = \hbar\omega + \bar\mu - E(P) + \Delta(P), \\
&\sigma_{22}(\boldsymbol{P},\omega) = \hbar\omega - \bar\mu + E(-P) - \Delta(-P), \\
&\sigma_{41}(\boldsymbol{P},\omega) = -\tilde\eta + U(P)\frac{\langle d(0)\rangle}{\sqrt{N}}, \\
&\sigma_{42}(\boldsymbol{P},\omega) = \tilde\eta - U(-P)\frac{\langle d(0)\rangle}{\sqrt{N}}, \\
&\sigma_{14}(\boldsymbol{P},\omega) = -2\tilde\eta, \quad \sigma_{24}(\boldsymbol{P},\omega) = 2\tilde\eta, \\
&\sigma_{44}(\boldsymbol{P},\omega) = \hbar\omega.
\end{aligned}
\tag{28}
$$

The dispersion laws for two exciton branches and the acoustical plasmon branch are

$$
\hbar\omega = \pm\sqrt{\left(\bar\mu - E(\boldsymbol{P}) + \Delta(0)\right)^2 + 4\tilde\eta\left(\tilde\eta - \frac{U(\boldsymbol{P})\langle d(0)\rangle}{\sqrt{N}}\right)},
\tag{29}
$$

$$
\hbar\omega_{AP}(P) = 0.
$$

In references [42,43] the coefficient $(B_{i-i} - 2A_{i-i} + \Delta(0))/I_l$ was determined to be $0.05r$. The rate r is $r = \dfrac{I_l}{\hbar\omega}$. The main parameters $(\bar{\mu} + \Delta(0))$, $\tilde{\eta}$ and $\dfrac{U(0)\langle d(0)\rangle}{\sqrt{N}I_l} = 2\dfrac{A_{i-i}uv}{I_l}$ are

$$(\bar{\mu} + \Delta(0)) = 0.1rv^2 I_l, \quad \tilde{\eta} = -0.1rv^3 I_l,$$

$$\frac{U(0)\langle d(0)\rangle}{\sqrt{N}} = 0.3ruv I_l, \quad u^2 + v^2 = 1. \quad (30)$$

The expression (29) was found as follows:

$$\frac{\hbar\omega}{I_l} = \pm\left(\left(0.1rv^2 - \frac{E(P)}{I_l}\right)^2 + 0.4rv^3\left(0.1rv^3 + 0.3ruv e^{-\frac{P^2 l^2}{2}}\right)\right)^{\frac{1}{2}}. \quad (31)$$

In the limit $P \to 0$ there is a gap in the energy spectrum

$$\hbar\omega_{ex}(0) = 2\sqrt{|\tilde{\eta}|\, U(0)\frac{\langle d(0)\rangle}{\sqrt{N}}} = 0.346rv^2\sqrt{u}I_l. \quad (32)$$

It depends on the Hartree term of the overall attractive interaction in the system proportional to $-U(P)$, with $U(P) > 0$, as well as on the quasiaverage theory parameter $\tilde{\eta}$ and on the amplitude of the condensate $\langle d(0)\rangle/\sqrt{N}$.

Unlike the case of a simple Bose-gas with repulsive interactions, the collective excitations of the magnetoexcitons in a BEC with $k = 0$ needs a finite amount of energy. The magnetoexciton subsystem is incompressible when only the excitons themselves are taken into account, and compressible when the optical plasmon branch is excited. In this approximation, the acoustical plasmon branch vanishes.

The energy spectrum described by the expressions (29) and (31) begins with a gap in the limit $P \to 0$, and has a roton-type behavior with a minimum in the point P_1, determined by the equality $E(P_1) = 0.1rv^2 I_l$ and the minimal value $\hbar\omega_{ex}(P_1) = I_l\sqrt{0.12r^2uv^4 e^{-P_1^2 l^2/2}}$. After the minimum the dispersion law transforms gradually in the energy spectrum of a free magnetoexciton. It is represented in Figure 4 for a specific value of the rate $r = 1/2$.

The acoustical plasmon branch has a dispersion law completely different from the optical plasmon oscillations. It has an absolute instability beginning with small values of wave vector going on up to the considerable value $pl \approx 2$. In this range of wave vectors, the optical plasmons have energies which do not exceed the activation energy $U(0)$. It means that the optical plasmons containing the opposite-phase oscillations of the electron and hole subsystems without displacement as a whole of their center of mass are allowed in the context of the attractive bath. On the other hand, the in-phase oscillations of the electron and hole subsystems in the composition of the
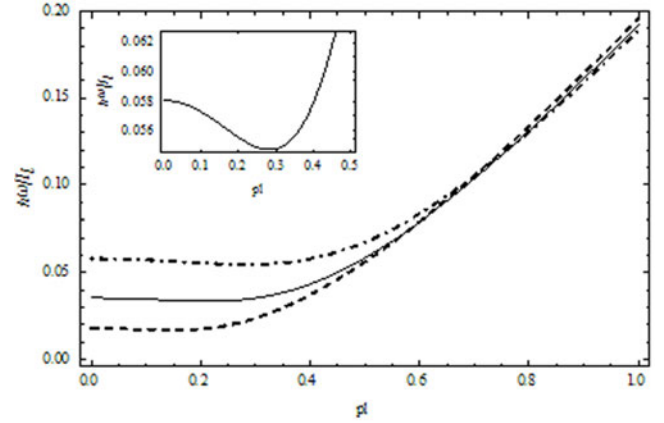

Fig. 4. The energy spectrum of the exciton branches in the case of BEC-ed magnetoexcitons with $\boldsymbol{k} = 0$ under influence of ELLs with parameter $r = 1/2$, without activation processes. Solid line corresponds energy spectrum with filling factor $\nu = v^2 = 1/5$, dashed line corresponds the filling factor $\nu = v^2 = 1/10$, and dot-dashed line is $\nu = v^2 = 1/3$.
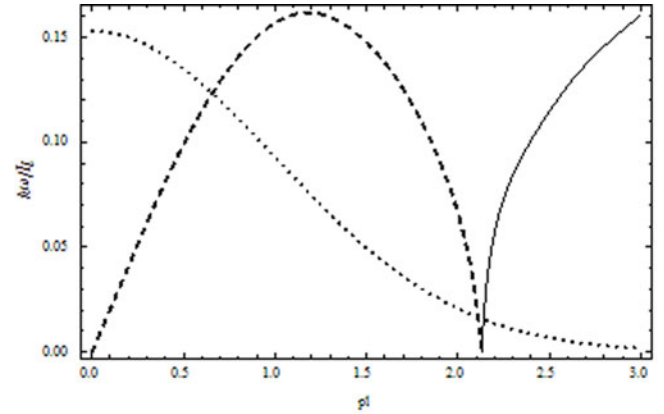

Fig. 5. The dispersion law of the acoustical plasmon branch. The solid line represents the real part, and the dashed line corresponds to the imaginary part. The dotted line gives the value $U(P)$.

acoustical plasmons are related to the displacements of their center of mass. Such displacements can take place only if their energy exceeds the activation energy $U(P)$. As a result, the acoustical plasmon branch has an imaginary part represented by the dashed line and is completely unstable in the region of wave vectors $pl \leq 2$. At greater values $pl > 2$ the energy spectrum is real and nonzero, approaching to the energy spectrum of the optical plasmons. The influence of the attractive bath $U(P)$ is represented by the dotted line. It vanishes in the limit $p \to \infty$. These properties of the acoustical plasmon branch are reflected in Figure 5.

The starting Hamiltonian (8) has two continuous symmetries. One is the gauge global symmetry $U(1)$ and another one is the rotational symmetry $SO(2)$. The resultant symmetry is $U(1) \times SO(2)$. The gauge symmetry is generated by the operator $\widehat{N}$ of the full particle number, when it commutes with the Hamiltonian. It means that the Hamiltonian is invariant under the unitary

transformation $\widehat{U}(\varphi)$ as follows

$$\widehat{U}(\varphi)\widehat{H}\widehat{U}^{-1}(\varphi) = \widehat{H}, \ \widehat{U}(\varphi) = e^{i\widehat{N}\varphi}, \ [\widehat{H}, \widehat{N}] = 0. \quad (33)$$

The operator $\widehat{N}$ is named as symmetry generator. The rotational symmetry $SO(2)$ is generated by the rotation operator $\widehat{C}_z(\varphi)$ which rotates the in-plane wave vectors $\overrightarrow{Q}$ on the arbitrary angle φ around z axis, which is perpendicular to the layer plane and is parallel to the external magnetic field. Coefficients $W_{\overrightarrow{Q}}$, $U(\overrightarrow{Q})$, $V(\overrightarrow{Q})$ in the formulas (4) and (7) depend on the square wave vector $\overrightarrow{Q}$ which is invariant under the rotations $\widehat{C}_z(\varphi)$. This fact determines the symmetry $SO(2)$ of the Hamiltonian (8). The gauge symmetry of the Hamiltonian (8) after the phase transition to the Bose-Einstein condensation (BEC) state is broken as it follows from expression (14). In the frame of the Bogoliubov theory of quasiaverages it contains a supplementary term proportional to $\widetilde{\eta}$. The gauge symmetry is broken because this term does not commute with the operator $\widehat{N}$. More so, this term is not invariant under the rotations $\widehat{C}_z(\varphi)$, because the in-plane wave vector $\overrightarrow{k}$ of the BEC is transformed into another wave vector rotated by the angle φ in comparison with the initial position. The second continuous symmetry is also broken. In such a way the installation of the Bose-Einstein condensation state with arbitrary in-plane wave vector $\overrightarrow{k}$ leads to the spontaneous breaking of the both continuous symmetries. We will discuss the more general case $\overrightarrow{k} \neq 0$ considering the case $\overrightarrow{k} = 0$ as a limit $\overrightarrow{k} \to 0$ of the cases with small values $kl \ll 1$. One can remember, that the supplementary terms in the Hamiltonian (8) describing the influence of the ELLs are actual in the range of small values $kl < 0.5$. Above we established that the number of the broken generators (BGs) denoted as N_{BG} equals to two ($N_{BG} = 2$).

The Goldstone theorem [84] states that the breaking of the continuous symmetry of the system leads to the appearance in the energy spectrum of the collective elementary excitations of the gapless branch equivalent to the massless particle in the relativistic physics. It happens because the system with continuous symmetry has initially a set of the degenerate minimal values of the potential energy leading to a set of degenerate vacuum states. For example, the dependence of the potential energy on the order parameter may be Mexican hatlike. The selection of one single vacuum state among the manifold of vacua and the fixing of the order parameter phase takes place due to quantum fluctuations and breaks spontaneously the continuous symmetry. The excitations over the selected vacuum transferring the system to the adjacent vacua and changing only the phase of the order parameter without change of its absolute value does not need any energy in the long wavelengths limit. Just these circumstances lead to the appearance of the gapless dispersion laws. These branches of the collective elementary excitations are named as the Nambu-Goldstone (NG) modes [84–86]. They are of two types. One of them, of the first-type has a linear (or odd) dispersion law in the range of the small

wave vectors, whereas the second-type has a quadratic (or even) dependence on the wave vector in the same region. The number of the NG modes in the system with many broken continuous symmetries was determined by the Nielsen and Chadha theorem [87]. It states that the number of the first-type NG modes N_I being accounted once and the number of the second type NG modes N_{II} being accounted twice equals or prevails the number of broken generators N_{BG}. It looks as follows $N_I + 2N_{II} \succeq N_{BG}$. In our case the optical plasmon branch has the properties of the second-type NG modes. We have $N_I = 0$, $N_{II} = 1$ and $N_{BG} = 2$. It leads to the equality $2N_{II} = N_{BG}$. The Goldstone theorem guarantees that the NG modes do not acquire mass at any order of quantum corrections. Nevertheless, sometimes soft modes appear, which are massless in the zeroth order, but become massive due to quantum corrections. They were introduced by Weinberg [88], who shown that such modes emerge if the symmetry of an effective potential of zeroth order is higher than that of the gauge symmetry. Following reference [89] now these modes are named as the quasi-Nambu-Goldstone modes, in spite of the fact that their initial name proposed by Weinberg was pseudo-NG modes. Georgi and Pais [90] demonstrated that the quasi-NG modes also occur in the cases in which the symmetry of the ground state is higher than that of the Hamiltonian.

The authors of reference [89] underlined that the spinor BEC are ideal systems to study the physics of the quasi-Nambu-Goldstone (NG) modes, because these systems have a great experimental manipulability and well established microscopic Hamiltonian. In reference [89] was shown that the quasi-NG modes appears in a spin-2 nematic phase. The ground state symmetry of the nematic phase at zeroth order approximation is broken by quatum corrections, thereby making the quasi-NG modes massive. Returning to the case of 2D magnetoexcitons in the BEC state with small wave vector $kl < 0.5$ described by the Hamiltonian (14), one can remember that the both continuous symmetries usual for the initial form (8) are lost. It happened due to the presence of the term $\widetilde{\eta}(d^{\dagger}_{\overrightarrow{k}} + d_{\overrightarrow{k}})$ in the frame of the Bogoliubov theory of the quasiaverages. Nevertheless the energy of the ground state as well as the self-energy parts $\Sigma_{ij}(P, \omega)$ were calculated only in the simplest case of the condensate wave vector $\overrightarrow{k} = 0$. These expressions can be relevant also for infinitesimal values of the modulus $|\mathbf{k}|$ but with a well defined direction. In this case the symmetry of the ground state will be higher than that of the Hamiltonian (14), what coincides with the situation described by Georgi and Pais [90]. It is one possible explanation of the quasi-NG modes appearance in the case of exciton branches of the spectrum.

Another possible mechanism of the gapped modes appearance is the existence of the local gauge symmetry, the breaking of which leads to the Higgs effect [91]. Such gauge transformation is a powerfull instrument revealing the fundamental physical processes. In the case of the fractional quantum Hall effects (FQHE) in the two-dimensional electron gas (2DEG) with fractional integer filling factor $\nu = 1/m, m > 1$ the breaking of the local gauge symmetry
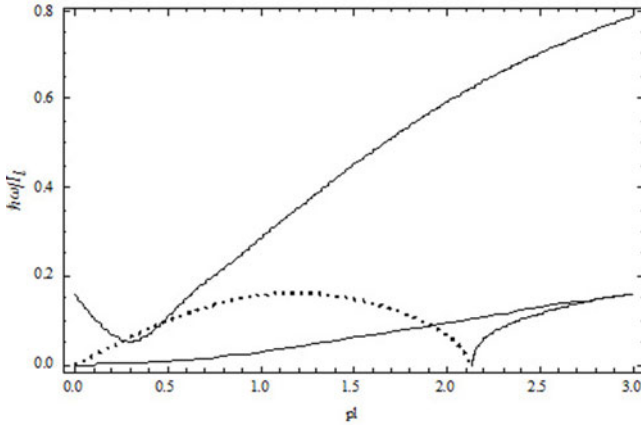
Fig. 6. Three branches of the collective elementary excitations: the exciton-type quasi-NG mode with a gap in the point $pl = 0$; the second-type NG mode describing the optical plasmons and the first-type NG mode with absolute instability (dotted line) describing the acoustical plasmons.

reveals the existence of the quantum vortices created under the influence of the magnetic flux quanta. In these condition each electron is attached by m quantum vortices forming together the composite fermion or boson. The interaction of the electrons with the attached vortices gives rise to a gapped energy spectrum of the collective elementary excitations as was established in references [19–21]. The applicability of these considerations to the case of the magnetoexcitons with arbitrary small filling factor $\nu < 1$ is not clear yet. The three branches of the energy spectrum are represented together in Figure 6. One of then is a second-type Nambu-Goldstone(NG) mode describing the optical plasmon-type excitations, the second branch is the first-type NG mode with absolute instability describing the acoustical-type excitations and the third branch is the quasi-NG mode describing the exciton-type collective elementary excitations of the system. The comparison of our result concerning the BEC of the coplanar magnetoexcitons with the ones obtained by Fertig [44] in the case of BEC of the quantum Hall excitons is instructive. In both models there is only one gapless Nambu-Goldstone mode between four branches of the energy spectrum. In our model it is related with the optical plasmon branch, whereas in the case of QHExs this mode is represented by the superposition of the operators describing the optical plasmon and exciton modes. In both models the exciton branches of the spectrum are not gapless and differ from the NG modes. In our case the exciton energy and quasienergy branches corresponding to normal and abnormal Green's functions have a gaps in the point $k = 0$, a roton-type segments in the region of intermediary wave vectors $kl \sim 1$ and saturation-type behaviors at great values of kl. In the case of reference [44] the exciton-type response function $\chi_z(q, \omega)$ and the acoustical-type response function $\chi_F(q, \omega)$ have no poles in the region of small energies in the frame of the LLLs. It was concluded that the energies of these excitations may be situated at greater values. In our case the acoustical plasmon branch reveals
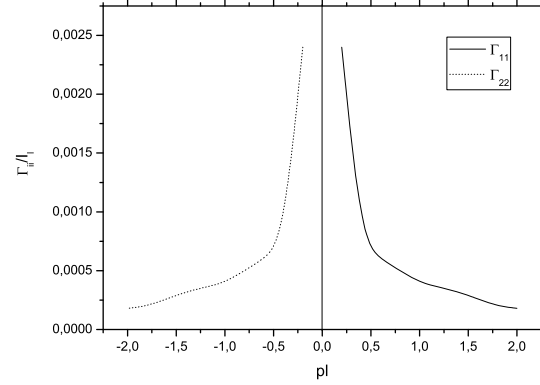

Fig. 7. The damping rates $\Gamma_{11}(\boldsymbol{P}, \omega)$ and $\Gamma_{22}(\boldsymbol{P}, \omega)$ of the diagonal self-energy parts.

an absolute instability in the range of small and intermediary values of k. It means that a real values of the pole does not exist in the range of small energies, which is similar with the results of reference [44]. One can conclude that the qualitative properties of the energy spectrum in both models are similar in spite of the mentioned differences. It is an additional argument supporting the accuracy of our calculations. We now discuss the damping rates of the obtained solutions. The damping rates of the obtained energy branches are determined by the imaginary parts of the self-energy parts

$$\Sigma_{ij}(\boldsymbol{P}, \omega) = \sigma_{ij}(\boldsymbol{P}, \omega) + i\Gamma_{ij}(\boldsymbol{P}, \omega). \qquad (34)$$

In the case of diagonal self-energy parts they are

$$\Gamma_{11}(\boldsymbol{P}, \omega) = \Gamma_{22}(-\boldsymbol{P}, -\omega)$$

$$= \frac{\langle D(P)D(-P) \rangle}{N^2} \pi \sum_Q U^2(Q) \cos^2\left(\frac{[P \times Q]_z l^2}{2} \right)$$

$$\times \delta\left(\hbar\omega + \bar{\mu} - E(P - Q) + \Delta(P - Q) \right),$$

$$\Gamma_{33}(\boldsymbol{P}, \omega) = \Gamma_{44}(\boldsymbol{P}, \omega) = 0. \qquad (35)$$

The damping rates $\Gamma_{11}(\boldsymbol{P}, \omega)$ and $\Gamma_{22}(\boldsymbol{P}, \omega)$ are nonzero in complementary regions of the frequencies and wave vectors and can be calculated using the zero order dependencies $\omega_{ex}(P)$ represented in Figure 3 without taking into account the fine details revealed in Figure 4.

The absolute values of the damping rates are drown in Figure 7. They are smaller than the corresponding real parts represented in Figure 4, which means that the obtained results have a physical sense. The damping rate of the optical plasmon branch in our description equals zero.

6 Conclusions

The energy spectrum of the collective elementary excitations of a 2D electron-hole (e-h) system in a strong

perpendicular magnetic field in a state of Bose-Einstein condensation (BEC) with wave vector $\boldsymbol{k} = 0$ has been investigated in the framework of Bogoliubov theory of quasi-averages. The starting Hamiltonian describing the e-h system contains not only the Coulomb interaction between the particles lying on the lowest Landau levels, but also a supplementary interaction due to their virtual quantum transitions from the LLLs to the excited Landau levels and back. This supplementary interaction generates, after the averaging on the ground state BCS-type wave function, direct Hartree-type terms with an attractive character, exchange Fock-type terms giving rise to repulsion, and similar terms arising after the Bogoliubov u-v transformation. The interplay of these three parameters gives rise to the resulting nonzero interaction between the magnetoexcitons with wave vector $\boldsymbol{k} = 0$ and to stability of their BEC.

The equations of motion for the exciton operators $d(P)$ and $d^\dagger(P)$ are interconnected with equations of motion for the density operators $\rho(P)$ and $D(P)$. Instead of a set of two equations of motion as in the case of usual Bose-gas, corresponding to normal and abnormal Green's functions, we have a set of four equations of motion. The change of the center-of-mass wave vector of the magnetoexciton, for example from 0 to $\overrightarrow{P}$, means the change of its internal structure because the internal distance between the Landau orbits of the quantized electron and hole becomes equal to $\left|\overrightarrow{P}\right| l^2$.

The separated electrons and holes remaining on their Landau orbits can take part in the formation of magnetoexcitons as well as in collective plasma oscillations. Such possibilities were not taken into consideration in the theory of structureless bosons or in the case of Wannier-Mott excitons with a rigid relative electron-hole motion structure without the possibility of the intra-series excitations. Magnetoexcitons have an internal structure that is much less rigid than standard Wannier-Mott excitons and the possibilities for electrons and holes to take part simultaneously in many processes are much more diverse. Instead of the branches of the energy spectrum corresponding to normal and abnormal Green's functions, we have to deal simultaneously with four branches of the energy spectrum, the two supplementary branches being the optical plasmon branch represented by the operator $\rho(P)$ and the acoustical plasmon branch represented by the operator $D(P)$.

The energy spectrum of the collective elementary excitations consists of four branches. Two of them are excitonic-type branches, one of them being the usual energy branch whereas the second one is the quasienergy branch representing the mirror reflection of the energy branch. The other two branches are the optical and acoustical plasmon branches. The exciton energy branch has an energy gap due to the attractive interaction terms, which needs to be overcome for excitation, as well as a roton-type region in the range of intermediary values of the wave vectors. At higher values of wave vector its dispersion law tends to saturation. The optical plasmon dispersion law is gapless with quadratic dependence in the range of small wave vectors and with saturation-type dependence in the

remaining part of the spectrum. The acoustical plasmon branch reveals an absolute instability of the spectrum in the range of small and intermediary values of the wave vectors. In the remaining range of the wave vectors the acoustical plasmon branch has a real value of the energy spectrum approaching the energy spectrum of the optical plasmon branch in the limiting case of great wave vectors.

In both models discussed above related with coplanar magnetoexcitons and with QHExs, there is only one gapless Nambu-Goldstone mode between four branches of the energy spectrum. In our model it is related with the optical plasmon branch, whereas in the case of QHExs this mode is represented by the superposition of the operators describing the optical plasmon and exciton modes. In both models the exciton branches of the spectrum are not gapless and differ from the NG modes. In our case the exciton energy and quasienergy branches corresponding to normal and abnormal Green's functions have a gaps in the point $k = 0$, a roton-type segments in the region of intermediary wave vectors $kl \sim 1$ and saturation-type behaviors at great values of kl. In the case of reference [44] the exciton-type response function $\chi_z(q,\omega)$ and the acoustical-type response function $\chi_F(q,\omega)$ have no poles in the region of small energies in the frame of the LLLs. It was concluded that the energies of these excitations may be situated at greater values. In our case the acoustical plasmon branch reveals an absolute instability in the range of small and intermediary values of k. It means that a real values of the pole does not exist in the range of small energies, which is similar with the results of reference [44]. One can conclude that the qualitative properties of the energy spectra in both models are similar in spite of the mentioned differences. It is an additional argument supporting the accuracy of our calculations, which satisfy to the Nielsen and Chadha theorem [87].

References

1. E. Prange, S.M. Girvin, *The Quantum Hall Effect* (Springer-Verlag, New York, 1986), p. 419
2. S. Das Sarma, A. Pinczuk, *Perspectives in Quantum Hall Effects* (John Wiley & Sons, Inc., New York, 1997), p. 430
3. H.L. Stormer, Rev. Mod. Phys. **71**, 875 (1999)
4. H. Enger, *Vortices in Chern-Simons-Ginzburg-Landau Theory and the Fractional quantum Hall Effect*, thesis (University of Oslo, 1998)
5. E.I. Rashba, Pure Appl. Chem. **67**, 409 (1995)
6. D.C. Tsui, H.L. Stormer, A.C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982)
7. R.B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983)

8. A.H. MacDonald, E.H. Rezayi, D. Keller, Phys. Rev. Lett. **68**, 1939 (1992)
9. F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982)
10. F. Wilczek, Phys. Rev. Lett. **49**, 957 (1982)
11. J.K. Jain, *Composite Fermions* (Cambridge University Press, Cambridge, 2007)
12. N. Read, Phys. Rev. Lett. **62**, 86 (1989)
13. N. Read, Semicond. Sci. Technol. **9**, 1859 (1994)
14. N. Read, `arXiv:cond-mat/9501090v1` (1995)
15. N. Read, Phys. Rev. B **58**, 16262 (1998)
16. B.I. Halperin, P.A. Lee, N. Read, Phys. Rev. B **47**, 7312 (1993)
17. S.M. Girvin, *The Quantum Hall Effect: Novel Excitations and Broken-Symmetries* (Indiana University, USA, 1998)
18. S.M. Girvin, A.H. MacDonald, Phys. Rev. Lett. **58**, 1252 (1987)
19. S.C. Zhang, T.H. Hansson, S. Kivelson, Phys. Rev. Lett. **62**, 82 (1989)
20. S.M. Girvin, A.H. MacDonald, P.M. Platzman, Phys. Rev. Lett. **54**, 581 (1985)
21. S.M. Girvin, A.H. MacDonald, P.M. Platzman, Phys. Rev. B **33**, 2481 (1986)
22. R.B. Laughlin, Rev. Mod. Phys. **71**, 863 (1999)
23. S. Kivelson, C. Kallin, D.P. Arovas, J.R. Schrieffer, Phys. Rev. Lett. **56**, 873 (1986)
24. D.H. Lee, G. Baskaran, S. Kivelson, Phys. Rev. Lett. **59**, 2467 (1987)
25. R.B. Laughlin, Phys. Rev. Lett. **60**, 2677 (1988)
26. S.S. Chern, J. Simons, Proc. Natl. Acad. Sci. **68**, 791 (1971)
27. D.H. Lee, S.C. Zhang, Phys. Rev. Lett. **66**, 1220 (1991)
28. C. Kallin, B.I. Halperin, Phys. Rev. B **30**, 5655 (1984)
29. R. Jackiw, P.S. Young, Phys. Rev. D **42**, 3500 (1990)
30. L.K. Myklebust, *Quantized Vortices* Cand. Scient. thesis (University of Oslo, 1996)
31. I.V. Lerner, Y.E. Lozovik, Zh. Eksp. Teor. Fiz. **78**, 1167 (1980)
32. I.V. Lerner, Y.E. Lozovik, J. Low Temp. Phys. **38**, 333 (1980)
33. I.V. Lerner, Y.E. Lozovik, Zh. Eksp. Teor. Fiz. **80**, 1488 (1981) [Sov. Phys. JETP **53**, 763 (1981)]
34. A.B. Dzyubenko, Y.E. Lozovik, Fiz. Tverd. Tela (Leningrad) **25**, 1519 (1983) [**25**, 874 (1983)]
35. A.B. Dzyubenko, Y.E. Lozovik, Fiz. Tverd. Tela (Leningrad) **26**, 1540 (1984) [**26**, 938 (1984)]
36. A.B. Dzyubenko, Y.E. Lozovik, Fiz. Tverd. Tela (Leningrad) J. Phys. A **24**, 415 (1991)
37. B. Paquet, T.M. Rice, K. Ueda, Phys. Rev. B **32**, 5208 (1985)
38. T.M. Rice, D. Paquet, K. Ueda, Helv. Phys. Acta **58**, 410 (1985)
39. S.A. Moskalenko, M.A. Liberman, D.W. Snoke, V. Botan, Phys. Rev. B **66**, 245316 (2002)
40. S.A. Moskalenko, M.A. Liberman, D.W. Snoke, V. Botan, B. Johansson, Physica E **19**, 278 (2003)
41. V. Botan, M.A. Liberman, S.A. Moskalenko, D.W. Snoke, B. Johansson, Physica B **346-347**, 460 (2004)
42. S.A. Moskalenko, M.A. Liberman, P.I. Khadzhi, E.V. Dumanov, I.V. Podlesny, V. Botan, Sol. State Commun. **140**, 236 (2006)
43. S.A. Moskalenko, M.A. Liberman, P.I. Khadzhi, E.V. Dumanov, I.V. Podlesny, V. Botan, Physica E **39**, 137 (2007)
44. H.A. Fertig, Phys. Rev. B **40**, 1087 (1989)
45. A.D.K. Finck, J.P. Eisenstein, L.N. Pfeiffer, K.W. West, Phys. Rev. Lett. **104**, 016801 (2010)
46. I.B. Spielman, J.P. Eisenstein, L.N. Pfeiffer, K.W. West, Phys. Rev. Lett. **84**, 5808 (2000)
47. Yafis Barlas, R. Cote, J. Lambert, A.H. MacDonald, Phys. Rev. Lett. **104**, 096802 (2010)
48. P. Giudici, K. Muraki, N. Kumada, T. Fujisawa, Phys. Rev. Lett. **104**, 056802 (2010)
49. D. Basu, L.F. Register, A.H. MacDonald, S.K. Banerjee, Phys. Rev. B **84**, 035449 (2011)
50. V.L. Berezinskii, Soviet Phys. JETP **67**, 493 (1971)
51. J.M. Kosterlitz, D.J. Thouless, J. Phys. C: Solid State Phys. **5**, L124 (1972)
52. J.M. Kosterlitz, D.J. Thouless, J. Phys. C: Solid State Phys. **6**, 1181 (1973)
53. W. Ketterle, N.J. van Drutten, Phys. Rev. A **54**, 656 (1996)
54. Y.E. Lozovik, O.L. Berman, Zhurn. Eksp. Teor. Fizik. **111**, 1879 (1997)
55. S. Stock, Z. Hadzibabic, B. Battelier, M. Cheneau, J. Dalibard, Phys. Rev. Lett. **95**, 190403 (2005)
56. T.P. Simula, M.D. Lee, D.A.W. Hutchinson, Phil. Mag. Lett. **85**, 395 (2000)
57. Z. Voros, R. Balili, D.W. Snoke, L. Pfeiffer, K. West, Phys. Rev. Lett. **94**, 226401 (2005)
58. Z. Voros, D.W. Snoke, L. Pfeiffer, K. West, Phys. Rev. Lett. **97**, 016803 (2006)
59. Z. Voros, D.W. Snoke, L. Pfeiffer, K. West, Phys. Rev. Lett. **103**, 016403 (2009)
60. S. Denev, V. Negoita, D.W. Snoke, B. Laikhtman, K. Eberl, L. Pfeiffer, Phys. Rev. B **66**, 205304 (2002)
61. M. Stern, V. Garmider, V. Umansky, I. Bar-Joseph, Phys. Rev. Lett. **100**, 256402 (2008)
62. D. Semkat, F. Richter, D. Kremp, G. Manzke, W.-D. Kraeft, K. Henneberger, Phys. Rev. B **80**, 155201 (2009)
63. D.W. Snoke, Solid State Commun. **146**, 73 (2008)
64. V. Negoita, D.W. Snoke, K. Eberl, Solid State Commun. **113**, 437 (2000)
65. N.W. Sinclair, J.K. Wuenschell, Z. Voros, B. Nelsen, D.W. Snoke, M.H. Szymanska, A. Chin, J. Keeling, L.N. Pfeiffer, K.W. West, Phys. Rev. B **83**, 245304 (2011)
66. V. Negoita, D.W. Snoke, K. Eberl, Appl. Phys. Lett. **75**, 2059 (1999)
67. D.W. Snoke, `arXiv:1208.1213v1[cond.mat.quant-gas]` (2012)
68. M. Combescot, O. Betbeder-Matibet, R. Combescot, Phys. Rev. Lett. **99**, 176403 (2007)
69. M. Combescot, M.N. Leuenberger, Solid State Commun. **149**, 13 (2009)
70. T. Hakioglu, M. Sahin, Phys. Rev. Lett. **98**, 166405 (2007)
71. M.A. Can, T. Hakioglu, Phys. Rev. Lett. **103**, 086404 (2009)
72. L.V. Keldysh, A.N. Kozlov, Zh. Eksp. Teor. Fiz. **54**, 978 (1968) [Sov. Phys. JETP **27**, 52 (1968)]
73. N.N. Bogoliubov, *Selected Works* (Gordon and Breach Sci. Publ., New-York, 1990), Vols. 1, 2
74. S.A. Moskalenko, M.A. Liberman, E.V. Dumanov, S.S. Rusu, I. Sanduleac, Mold. J. Phys. Sci. **9**, 16 (2010)
75. S.A. Moskalenko, M.A. Liberman, E.V. Dumanov, J. Nanoelectron. Optoelectron. **4**, 52 (2009)
76. A.A. Abrikosov, L.P. Gorkov, I.E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics*, edited by R.A. Silverman (Dover, New York, 1975), p. 444

77. D.N. Zubarev, Sov. Phys. Uspekhi Fiz. Nauk **71**, 71 (1960)
78. D. Pines, in *Elementary Excitations in Solids*; edited by W.A. Benjamin (New York, 1963)
79. T.M. Rice, in *Solid State Physics*, edited by H. Ehrenreich, F. Seitz, D. Turnbull (Academic press, New York, 1977), p. 32
80. S.A. Moskalenko, D.W. Snoke, *Bose-Einstein Condensation of Excitons and Biexcitons and Coherent Nonlinear Optics with Excitons* (Cambridge University Press, Cambridge UK, New York, USA, 2000), p. 415
81. S.T. Beliaev, Zh. Eksp. Teor. Fiz. **34**, 417 (1958) [Sov. Phys. JETP **7**, 299 (1958)]
82. S. Das Sarma, A. Madhukar, Phys. Rev. B **23**, 805 (1981)
83. S.A. Moskalenko, M.A. Liberman, E.V. Dumanov, A.G. Stefan, M.I. Shmiglyuk, J. Phys.: Condens. Matter **21**, 235801 (2009)
84. J. Goldstone, Nuovo Cimento **19**, 154 (1961)
85. Y. Nambu, Phys. Rev. **117**, 648 (1960)
86. Y. Nambu, Phys. Rev. Lett. **4**, 380 (1960)
87. H.B. Nielsen, S. Chadha, Nucl. Phys. B **105**, 445 (1976)
88. S. Weinberg, Phys. Rev. Lett. **29**, 1698 (1972)
89. S. Uchino, M. Kobayashi, N. Nitta, M. Ueda, Phys. Rev. Lett. **105**, 230406 (2010)
90. H. Georgi, A. Pais, Phys. Rev. D **12**, 508 (1975)
91. L.H. Ryder, *Quantum Field Theory* (Cambridge University Press, 1985)