Gravitation & Cosmology, Vol. 0 (2005), No. 0 (00), pp. 1–12

Diffusion model of evolution of superthermal high-energy particles under scaling in early Universe

Yu.G.Ignat'ev, R.A.Ziatdinov

Kazan State Pedagogical University,
Mezhlauk str., 1, Kazan 420021, Russia

Evolution of superthermal relict component is research on basis of non-equilibrium model of Universe and kinetic equation of Fokker-Planck type offered by one of the Authors.

Диффузионная модель эволюции сверхтепловых высокоэнергетических частиц при наличии скейлинга в расширяющейся Вселенной
Ю.Г.Игнатьев, Р.А.Зиатдинов

На основе предложенных одним из Авторов неравновесной модели Вселенной и кинетического уравнения типа Фоккера-Планка изучается эволюция сверхтепловой реликтовой компоненты.

1. Introduction

Relativistic kinetic equation respect to macroscopic distribution function
$f_a(x^i, p^k)$ a type particles [1]-[4]:

$$p^i \tilde{\nabla}_i f_a(x,p) = \sum_{b,c,d} J_{ab\leftrightarrows cd}(x,p), \qquad (1)$$

where $\tilde{\nabla}$ - Cartans' covariant differentiation operator in phase space $X \times P$:

$$\tilde{\nabla} = \nabla_i + \Gamma^j_{ik} p^k \frac{\partial}{\partial p^j}. \qquad (2)$$

Using distribution function $f_a(x,p)$ macroscopic moments are determined:

$$n^i_a(x) = \int\limits_{P(x)} f_a(x,p) p^i dP, \qquad (3)$$

- density vector of a type particles flux and

$$T^{ik}_a(x) = \int\limits_{P(x)} p^i p^k f_a(x,p) dP, \qquad (4)$$

- a type particles energy-momentum tensor, where

$$dP = \sqrt{-g} d^3p/p^4 \qquad (5)$$

- invariant element of momentum space volume. Reducing formula (4) by means of metric tensor g_{ik} and taking into account the 4-momentum normalization relation:

$$(p,p) = m_a^2, \qquad (6)$$

we get:

$$T^a_S(x) = m_a^2 \int\limits_{P(x)} f_a(x,p) dP, \qquad (7)$$

where $T^a_S(x)$ - trace of a type particles energy-momentum tensor.

In case of homogeneous isotropic distribution $f(\eta, p)$ in Freedmans' metric:

$$ds^2 = a^2(\eta)(d\eta^2 - dl^2) = dt^2 - a^2(t) dl^2, \qquad (8)$$

where:

$$dl^2 = d\chi^2 + \rho^2(\chi) d\Omega^2, \qquad (9)$$

$$\rho(\chi) = \begin{cases} \mathrm{sh}(\chi), & k = -1; \\ \chi, & k = 0; \\ \sin(\chi), & k = +1 \end{cases},$$

k -index of three-dimensional space curvature, kinetic equations taking the form:

$$\frac{\partial f_a}{\partial t} - \frac{\dot{a}}{a} p \frac{\partial f_a}{\partial p} = \frac{1}{\sqrt{m_a^2 + p^2}} \sum_{b,c,d} J_{ab\leftrightarrows cd}(t,p), \qquad (10)$$

or in η, p variables:

$$\frac{\partial f_a}{\partial \eta} - \frac{a'}{a} p \frac{\partial f_a}{\partial p} = \frac{a(\eta)}{\sqrt{m_a^2 + p^2}} \sum_{b,c,d} J_{ab\leftrightarrows cd}(t,p), \qquad (11)$$

where $\dot{a}$-derivative with respect to time t and a'-derivative with respect to time variable η, moreover:

$$a(\eta) d\eta = dt.$$

As shown in [5], that in ultrarelativistic limit under the condition of conformal invariance of non-gravitational macroscopic field equations and scale invariance of matrix elements of interaction (which vary only as a result of transformation of phase space volume element):

$$\overline{|M(p,q|p',q')|^2} = a^2(\eta)|M(p,q|p',q')|^2 \quad (12)$$

kinetic equations are conformally invariant. Then in conformally corresponding space we will obtain standard result of kinetic theory: local thermodynamic equilibrium restores at times much more than some effective times of interactions. It means that in Freedmans' space at early times local thermodynamic equilibrium must be absent. This article is dedicated to study of relaxation process to local thermodynamic equilibrium.

2. Four-piece reactions kinematics

Four-piece reactions such as:

$$a + b \to c + d \quad (13)$$

fully described by two kinematic invariants, s and t, which have following meaning: $\sqrt{s}$- energy of colliding particles in frame of mass center:

$$s = (p_a + p_b)^2 = (p_c + p_d)^2, \quad (14)$$

t-relativistic square of transmitted momentum:[1]

$$t = (p_c - p_a)^2 = (p_b - p_d)^2, \quad (15)$$

where momentum squares are understood as scalar four-piece squares:

$$p_a^2 = (p_a, p_a) = (p^4)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 = m_a^2,$$

etc. For example:

$$(p_a + p_b)^2 = p_a^2 + 2(p_a p_b) + p_b^2 = m_a^2 + 2(p_a, p_b) + m_b^2.$$

Invariant scattering amplitudes $F(s,t)$, determined as a result of averaging-out of invariant scattering amplitudes by particles state, c and d, turn out to be depending only on these two invariants (e.g., see [6]):

$$\sum |M_{FJ}|^2 = \frac{|F(s,t)|^2}{(2S_c+1)(2S_d+1)}, \quad (16)$$

where S_i - are spins. Using invariant amplitude $F(s,t)$ total crossection of reaction is determined (13) (see [6]):

$$\sigma_{tot} = \frac{1}{16\pi^2\lambda^2(s, m_a^2, m_b^2)} \int\limits_{t_{min}}^{0} dt |F(s,t)|^2, \quad (17)$$

[1]Authors hope that following notation coincidence will not confuse readers : t - time in Freedmans' metric, s - its interval, simultaneously t, s - kinematic invariants. This notation is standard and we didn't consider that it is necessary to change it.

where λ - triangle function:

$$\lambda(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc;$$

$$t_{min} = -\frac{\lambda^2}{s}.$$

In ultrarelativistic limit:

$$\frac{p_i}{m_i} \to \infty; \qquad \frac{s}{m_i^2} \to \infty \quad (18)$$

$\lambda \to s$, and formula (17) considerably simplifies introducing dimensionless variable:

$$x = -\frac{t}{s} : \quad (19)$$

$$\sigma_{tot} = \frac{1}{16\pi s} \int\limits_0^1 dx |F(s,x)|^2. \quad (20)$$

Let us consider four-piece reaction in the form of (13). Three-dimensional (trivariate) momentum modules of corresponding particles will be symbolized as p, q, p', q' for short. Let's introduce the angles: φ-angle between momentum vectors of particles and $\vec{q}$, ψ - angles between momentum vectors $\vec{p}\,'$ and $\vec{p}'$, φ' - angle between vectors $\vec{p}\,'$ and $\vec{q}\,'$. Conservation laws of four-dimensional momentum of particles we will write in the following way:

$$\sqrt{m_a^2 + p^2} + \sqrt{m_b^2 + q^2} = \sqrt{m_c^2 + p'^{\,2}} + \sqrt{m_d^2 + q'^{\,2}} \quad (21)$$

- law of conservation of energy and

$$\vec{p} + \vec{q} = \vec{p}\,' + \vec{q}\,', - \quad (22)$$

- law of conservation of momentum. Further find:

$$s = m_a^2 + m_b^2 + 2\sqrt{m_a^2 + p^2}\sqrt{m_b^2 + q^2} - 2pq\cos\varphi.$$

$$t = m_a^2 + m_c^2 - 2\sqrt{m_a^2 + p^2}\sqrt{m_c^2 + p'^2} + 2pp'\cos\psi.$$

In ultrarelativistic limit $p/m \to \infty$ get from (21):

$$p + q = p' + q'. \quad (23)$$

So, raising to the second power relations (22) and (23), we will get in ultrarelativistic limit:

$$pq(1 - \cos\varphi) = p'q'(1 - \cos\varphi'). \quad (24)$$

$$\lim_{p,q\to\infty} s = 2pq(1 - \cos\varphi) = 4pq\cos^2\frac{\varphi}{2};$$

$$\lim_{p,p'\to\infty} t = -2pp'(1 - \cos\psi) = -4pp'\cos^2\frac{\psi}{2}. \quad (25)$$

Hence, in ultrarelativistic limit for variable x we will obtain:

$$x = \frac{q(1 - \cos\varphi)}{p'(1 - \cos\psi)}. \quad (26)$$

Assumed that $p' = p - \Delta p$, $q' = q - \Delta q$, on account of law of conservation of energy (23) we will obtain:

$$\Delta q = -\Delta p,$$

hence:

$$p' = p - \Delta p; \qquad q' = q + \Delta p. \tag{27}$$

In ultrarelativistic limit it is possible to write variable Δp (18) in form of:

$$\Delta p = x(p-q) - \cos\varphi\sqrt{x(1-x)(4pq-s)}. \tag{28}$$

3. Scaling and asymptotic scattering crossection

For analysis of kinetics of processes in early Universe it is necessary to know asymptotic behavior of invariant amplitudes $F(s,t)$ in limit (18). Modern experimental opportunities have coefficient restriction $\sqrt{s}$ at degree of hundreds GeV. It would be risky to bear on that or other field model of interaction for prediction of asymptotic behavior of scattering crossection in the range of superhigh energies. It is more rational in recent conditions to bear on axiomatic theory of S-matrix conclusions get on basis of fundamental laws of unitarity, causality, scale invariance etc. Unitarity of S-matrix leads to well-known asymptotic relation (see, e.g., [8]):

$$\left.\frac{d\sigma}{dt}\right|_{s\to\infty} \sim \frac{1}{s^2} \tag{29}$$

for variables s higher than unitary limit, i.e., under the condition (18),if m_i means the masses of all intermediate particles. But from (20) results:

$$F(s,1)|_{s\to\infty} \sim \text{Const}. \tag{30}$$

In the sixties of XX century on basis of axiomatic theory of S-matrix were received stringent restrictions of asymptotic behavior of total crossections and invariant scattering amplitudes:

$$\frac{C_1}{s^2\ln s} < \sigma_{tot}(s) < C_2\ln^2 s, \tag{31}$$

where C_1, C_2 - unknown constants. Upper limit (31) was determined in works [9]-[11], lower limit - in [12], [13] (see also review in book [14]). We also notice restriction to invariant scattering amplitudes (see, e.g., [14]):

$$|F(s,t)| \leqslant |F(s,0)|; \tag{32}$$

$$C_1' < |F(s,0)| < C_2' s\ln^2 s. \tag{33}$$

Therefore, invariant scattering amplitudes in limit (18) must be functions of variable $x = -t/s$, i.e.:

$$|F(s,t)| = |F(x)|, \ (s\to\infty). \tag{34}$$

But in consequence of (20)

$$\sigma_{tot}(s) = \frac{1}{16\pi s}\int\limits_0^1 dx|F(x)|^2 = \frac{\text{Const}}{s}, - \tag{35}$$

total crossection behaves such as the crossection of electromagnetic interaction, i.e. scaling restores under superhigh energies. Scaling asymptotics of crossection (35) lies strictly between possible extreme asymptotics of complete scattering crossection (31). Moreover, under holding (35) automatically realize relations obtained on basis of axiomatic theory of S-matrix (29) and (30). Further, as described above, scaling exists for pure electromagnetic interactions in consequence of their scale invariance. For lepton-hadron interaction assumption of scaling existence was offered in works [15],[16]. Especially, for total crossection of reaction

$$e + e^+ \to \text{hadrons}$$

following expression was obtained:

$$\sigma_{tot} = \frac{4\pi\alpha^2}{3s}\sum e_i^2,$$

where α - fine structure constant, e_i - charges of fundamental fermion fields. Data, received on Stanford accelerator, verify existence of scaling for this interactions. Apparently, for gravitational interactions scaling also must restore under superhigh energies in consequence of scale invariance of gravitational interactions in WKB-approximation [18]. Later on we will assume existence of scaling under energies higher than unitary limit $s\to\infty$. The question arises about value of constant in formula (35) and logarithmic refinement of this constant. This value can be estimated from following consideration. If idea of union of all interactions on Planck energy scales $E_{pl} = m_{pl} = 1$, then for $s \sim 1$ all interactions must be described by single scattering crossection, formed from three fundamental constants $G, \hbar, c$, i.e., in chosen scale of units must be:

$$\sigma|_{s\sim 1} = \pi l_{pl}^2 \quad (=\pi). \tag{36}$$

However, in order that scattering crossection decreased to such values on Planck energy scales, starting from values of order $\sigma_T = 8\pi\alpha^2/3m_e^2$ (m_e - mass of electron, σ_T - Thompson scattering crossection) for electromagnetic interactions, i.e., for $s \sim m_e^2$, it must decrease inversely s, i.e., and what is more by scaling law .[2] Logarithmically correcting this relation, we introduce *universal asymptotic scattering crossection* (UACS), derived in papers [19], [20]:

$$\sigma_0(s) = \frac{2\pi}{s\left(1+\ln^2\frac{s}{s_0}\right)}, \tag{37}$$

[2]We note that this fact is one more independent reason in favour of scaling existence in the range of high energies.

where $s_0 = 4$ - square of total energy of two colliding Planck masses.

Scattering crossection σ_0, UACS derived by formula (37), has some unusual properties:

1. UACS formed by fundamental constants $G, \hbar, c$;
2. UACS behaves so that its values strictly lie between possible extreme limits of asymptotic behavior of scattering (31), determined by means of asymptotic theory of S-matrix;
3. UACS is a scaling scattering crossection with logarithmic accuracy;
4. For scattering reaction of photon by nonrelativistic electron ($s = m_e^2$) formula (37) gives $\sigma_0 = 4/3\sigma_T \sim \sigma_T$;
5. For electro-weak interactions ($s = m_W^2$, where m_w - mass of intermediate W-boson) under $\sin\theta_W = 0,22$ (see, e.g., [8]) we will get from (37) $\sigma_0 = 0,78\sigma_W$, where $\sigma_W = G_F^2 m_W^2/\pi$ - crossection of scattering νe taking into account intermediate W-boson;
6. Under Planck values of energy $\sigma_0(m_{pl}^2) \approx \sigma_{pl}$.

These unusual features of UACS and surprising coincidence of its values with crossections of well-known processes hardly can be occasional on wide ranges of energy values (from m_e to $10^{22}m_e$). This allows us to use UACS as an accurate formula for asymptotic value of scattering crossections for all interactions.

In case that from this point on we will discuss reactions kinetics only in range of superhigh energies, in which all interactions described as we suppose by UACS, no difference can be between particles in integrals of interactions, taking into account only there, where it is necessary their spin and other characteristics.

In this sense all interactions become uniform under superhigh energies and four-piece interactions efficiently described as elastic, what makes much easier to analyze these processes.

4. Derivation of collisions integral in Fokker-Planck form

Integral of elastic paired collisions for reactions of type (13) for isotropic distributions $f_a(p, x^i)$, depending on absolute value of momentum, can be reduced to form [7]:

$$J_{ab}(p) = \frac{2S_b+1}{(2\pi)^3 p}\int\limits_0^\infty \frac{qdq}{\sqrt{m_b^2+q^2}}\int\limits_0^{4pq}\frac{ds}{16\pi}\int\limits_0^1 dx\overline{|M(s,x|^2}\times$$

$$\int\limits_0^{2\pi} d\varphi\,[f_a(p')f_b(q') - f_a(p)f_b(q)]\,, \qquad (38)$$

where it is necessary to substitute expression (27) for p', q' and (28) for Δp.

Let's assume, as it frequently is done, that at collisions of particles on average small momentum is transmitted, i.e.,

$$\overline{(p_a - p_c)^2} \ll \overline{p^2}, \qquad (39)$$

to that values of variable $x \to 1$ correspond. Setting

$$x = 1 - \xi^2, \qquad (40)$$

Let's take Taylor of collisions integral in smallness of transmitted momentum, i.e., in smallness of parameter $\xi^2 \ll 1$.

It follows from formula (28) that:

$$\Delta p = (1-\xi^2)(p-q) - \cos\varphi\xi\sqrt{4pq-s}. \qquad (41)$$

Retaining members of degree ξ^2, we will write expansions of distribution functions:

$$f(p') = f(q) + \frac{df}{dq}[\cos\varphi\xi\sqrt{4pq-s} + \xi^2(p-q)] +$$
$$+\frac{1}{2}\frac{d^2f}{dq^2}\xi^2(4pq-s)\cos^2\varphi;$$
$$f(q') = f(p) - \frac{df}{dp}[\cos\varphi\xi\sqrt{4pq-s} + \xi^2(p-q)] +$$
$$+\frac{1}{2}\frac{d^2f}{dp^2}\xi^2(4pq-s)\cos^2\varphi.$$

By integration in angular variable, integrals linear on ξ, will vanish. Therefore, for interior integral we will get the following expression:

$$\frac{1}{2\pi}\int\limits_0^{2\pi} d\varphi\,[f(p')f(q') - f(p)f(q)] =$$

$$f(p)f(q) + (p-q)\xi^2\left[f(p)\frac{df(q)}{dq} - f(q)\frac{df(p)}{dp}\right] + \qquad (42)$$

$$\frac{1}{4}\xi^2(4pq-s)\left[f(p)\frac{d^2f(q)}{dq^2} - 2\frac{df(p)}{dp}\frac{df(q)}{dq} + f(q)\frac{d^2f(p)}{dp^2}\right].$$

By integrating in variables x assume that:

$$A = \int\limits_0^1 F(s,x)x(1-x)ds;\ B = \int\limits_0^1 xF(s,x)dx, \quad (43)$$

so that:

$$A + B = \int\limits_0^1 F(s,x)dx. \qquad (44)$$

Later on we will assume the fact, that scaling restores under superhigh energies, i.e. the relation (35) is realized, according to what $F(s,x) \approx F(x)$ so that $A \approx \mathrm{Const}, B \approx \mathrm{Const}$.

Then by integrating in variables s we will find in received relation:

$$J_{ab}(p) = A\frac{2S_b+1}{(4\pi)^3 p}\int\limits_0^\infty dq\left\{\left[f(p)\frac{df(q)}{dq} - f(q)\frac{df(p)}{dp}\right] + \right.$$

$$\left. + 2p^2q^2\left[f(p)\frac{d^2f(q)}{dq^2} - 2\frac{df(p)}{dp}\frac{df(q)}{dq} + f(q)\frac{d^2f(p)}{dp^2}\right]\right\}$$

Let's integrate by parts in the part of integral including second derivatives in variable q, in this case:

$$\int\limits_0^\infty q^2\frac{d^2f(q)}{dq^2}dq = q^2\left.\frac{df(q)}{dq}\right|_0^\infty - 2\int\limits_0^\infty q\frac{df(q)}{dq}dq =$$

$$= -2qf(q)|_0^\infty + 2\int\limits_0^\infty f(q)q^2dq = 2\int\limits_0^\infty f(q)q^2dq,$$

considering:

$$\lim_{q\to\infty} q^n f(q) = 0, \quad (0 \leqslant n \leqslant 3), \tag{45}$$

necessary condition for expressions convergence for numerical density of particles and energy.

So, integrating by parts, we finally get collisions integral in Fokker-Planck form:

$$J_{ab}(p) = A\frac{2S+1}{4(2\pi)^3 p}\times$$

$$\frac{\partial}{\partial p}\left[p^2\int\limits_0^\infty q^2\left(f(q)\frac{\partial f(p)}{\partial p} - f(p)\frac{\partial f(q)}{\partial q}\right)dq\right]. \tag{46}$$

5. Kinetic equation for superthermal component in diffusive approximation

Substituting obtained collisions integral in Fokker-Planck form (46) in kinetic equation (10), we get at ultrarelativistic values of impulse p:

$$\frac{\partial f_a}{\partial t} - \frac{\dot{a}}{a}p\frac{\partial f_a}{\partial p} = A\frac{2S+1}{4(2\pi)^3p^2}\times$$

$$\frac{\partial}{\partial p}\left[p^2\int\limits_0^\infty q^2\left(f(q)\frac{\partial f(p)}{\partial p} - f(p)\frac{\partial f(q)}{\partial q}\right)dq\right]. \tag{47}$$

Integrating by parts in equatuon (47) and taking into account relations:

$$n(t) = \frac{4\pi(2S+1)}{(2\pi)^3}\int\limits_0^\infty q^2 f(q)dq, \tag{48}$$

- numerical density of particles,

$$T_S(t) = \frac{4\pi(2S+1)}{(2\pi)^3}\int\limits_0^\infty qf(q)dq, \tag{49}$$

- trace of momentum-energy tensor of particles, equation (47) acquires the form:

$$\frac{\partial f_a}{\partial \eta} - \frac{a'}{a}p\frac{\partial f_a}{\partial p} =$$

$$\frac{A}{p^2}\frac{\partial}{\partial p}p^2\left[n(\eta)\frac{\partial f(p)}{\partial p} + 2T_S(\eta)f(p)\right] \tag{50}$$

In the framework considered models in ultrarelativistic area in consequence of particles identically conservation laws of number of particles, in result:

$$n(\eta)a^3(\eta) = \mathrm{Const} \Rightarrow n(\eta) = \frac{n_*}{a(\eta)^3}. \tag{51}$$

In universal system of units ($G = \hbar = c = 1$) we chose scale factor in the form of:

$$a(t) = \sqrt{2t} = \eta,$$

so that at half-Planck point in time ($\eta = 1, t = \frac{1}{2}$) we have $a = 1$, $n = n_0$, Let's pass to so-called *conformal momentum* $\mathcal{P}$, which is movement integral of free particles in Freedmans' metric, by formula (see, e.g., [1]):

$$p = a(\eta)\mathcal{P}, \tag{52}$$

so that at Planck point in time $p = \mathcal{P}$. Let's introduce according to (47),(48) and (52) conformal numerical density of particles and density of energy:

$$n_*(\eta) = \frac{4\pi(2S+1)}{(2\pi)^3}\int\limits_0^\infty f(\eta,\mathcal{P})\mathcal{P}^2d\mathcal{P} \quad (= \mathrm{Const} = n_0);$$

$$\varepsilon_*(\eta) = \frac{4\pi(2S+1)}{(2\pi)^3}\int\limits_0^\infty f(\eta,\mathcal{P})\mathcal{P}^3d\mathcal{P},$$

also introduce average conformal momentum (energy):

$$<\mathcal{P}> = \frac{\varepsilon_*(\eta)}{n_*(\eta)}, \tag{53}$$

so that:

$$n = \frac{n_*}{a^3}; \quad \varepsilon = \frac{\varepsilon_*}{a^4}; \quad \Rightarrow \varepsilon = <p> n, \tag{54}$$

where:

$$<p> = \frac{<\mathcal{P}>}{a}.$$

Relation for conformal densities is realized in consequence of (54):

$$\varepsilon_*(\eta) = <\mathcal{P}(\eta)> n_*. \tag{55}$$

Let's introduce dimensionless function $\beta(\eta)$ by means of relation:

$$\beta(\eta)n_* = \frac{4\pi(2S+1)}{(2\pi)^3}\int\limits_0^\infty f(\eta,\mathcal{P})\mathcal{P}d\mathcal{P}. \qquad (56)$$

By means of introduced notation we adduce equation (50) to more elegant form respect to function $f(\eta,\mathcal{P})$ (see, e.g., different variants [20], [21]):

$$\frac{\partial f}{\partial\eta} = \frac{An_*}{\mathcal{P}^2}\frac{\partial}{\partial\mathcal{P}}\mathcal{P}^2\left(\frac{\partial f}{\partial\mathcal{P}} + 2\beta(\eta)f\right), \qquad (57)$$

- this is unknown kinetic equation in diffusive approximation.

As is easy to see that in consequence of definition (56) function $\beta(\eta)$ is integral of distribution function. So, in spite of its outer simplicity, equation (57) remains integro-differential. As we see, ultrarelativistic equilib-
rium function of distribution is:

$$f_0 = C(\eta)e^{-2\dfrac{\beta(\eta)}{\mathcal{P}}}, \qquad (58)$$

where $C(\eta)$ - arbitrary function which converts to zero received collisions integral. It means that with time $\eta\to\infty$ solution of equation (57) approaches to equilibrium distribution (58) with temperature:

$$T(\eta) = \frac{\beta(\eta)}{a(\eta)} \Rightarrow T_* = \beta(\eta), \qquad (59)$$

where T_* - conformal temperature.

6. Evolution of superthermal particles in strongly non-equilibrium Universe

Let's consider the Universe at ultrarelativistic stage of expansion and $\mathcal{E}$ - total density of matter energy, $\mathcal{E}_*$ - its conformal value. Then from Einstein's equations follows:

$$\mathcal{E} = \frac{3}{32\pi t^2} = \frac{3}{8\pi a^4} = \frac{3}{8\pi\eta^4} \Rightarrow \mathcal{E}_* = \frac{3}{8\pi}. \qquad (60)$$

Let's consider the evolution of superthermal particles when number of equilibrium particles lower than unitary limit in energy range order, n_T much less than number of superthermal particles in scaling range,

$$n_T \ll n. \qquad (61)$$

In this case $\mathcal{E} = \varepsilon$, so that:

$$\varepsilon_* = \frac{3}{8\pi} = \text{Const}. \qquad (62)$$

For equilibrium distributions of ultrarelativistic particles:

$$f_T = \frac{1}{e^{\frac{p}{T}} \pm 1}, \qquad (63)$$

where $+1$ corresponds to fermions, -1 - to bosons, numerical density of particles (48) equals to:

$$n_T = \mu_n\frac{2S+1}{\pi^2}T^3\zeta(3), \qquad (64)$$

where statistical factor $\mu_n = 1$ - for bosons and $\mu_n = 3/4$ - for fermions, T - temperature. Appropriate equilibrium densities of energy equal to:

$$\varepsilon_T = \mu_\varepsilon\frac{2S+1}{30\pi^2}T^4, \qquad (65)$$

where statistical factor $\mu_\varepsilon = 1$ for bosons and $\mu_\varepsilon = 7/8$ for fermions. Total density of energy of equilibrium ultrarelativistic component equals to:

$$\varepsilon_T = \frac{g}{30\pi^2}T^4, \qquad (66)$$

where g - statistical factor:

$$g = \sum_B(2S+1) + \frac{7}{8}\sum_F(2S+1),$$

where summation is taken over by all thermal bosons and fermions.

But in consequence of energy conservation law and Einsteins equations following equality must realize:

$$\varepsilon_* + gT_*^4 = \frac{3}{8\pi}, \qquad (67)$$

where

$$T_*(\eta) = T/a(\eta).$$

Relation (67) can be considered as equation for determining temperature of equilibrium component (see details in [20], [21]).

In this article we will discuss evolution of superthermal component at the early stages of expansion, when condition $\varepsilon_T \ll \varepsilon$ is met , or what is same:

$$gT_*^4 \ll \frac{3}{8\pi}. \qquad (68)$$

Then

$$\varepsilon_* = \frac{3}{8\pi}, \qquad (69)$$

and in consequence of (51), (55) following relation is realized:

$$<\mathcal{P}(\eta)> = \frac{3}{8\pi n_*} = \text{Const} = \mathcal{P}_0, \qquad (70)$$

- on this stage the average value of conformal energy of superthermal particles doesn't change with time. For values $\mathcal{P}_0 \sim 1$ there are Planck energies on Planck times, i.e. energies order to thermal in hot model of Universe. Therefore we will set:

$$\mathcal{P}_0 \gg 1. \qquad (71)$$

But in consequence of (55)

$$n_* = \frac{3}{8\pi\mathcal{P}_0} \ll 1. \tag{72}$$

Making estimate of function $\beta(\eta)$ and taking into account reduced relations, we obtain:

$$\beta(\eta) \sim \frac{1}{\mathcal{P}_0} \ll 1. \tag{73}$$

In this article we assume for short

$$\beta(\eta) \approx \mathrm{Const}.$$

In case that dimensionless value $\mathcal{P}_0$ is free parameter of model, we pass to dimensionless momentum variable x:

$$x = \frac{\mathcal{P}}{\mathcal{P}_0}. \tag{74}$$

Substituting the variables, we get relation of normalization for distribution function from (54):

$$\int\limits_0^\infty f(\eta, x)x^2dx = \frac{3\pi}{4(2S+1)}\frac{1}{\mathcal{P}_0^4}. \tag{75}$$

Thus, order of distribution function is

$$f(\eta, x) \sim \frac{1}{\mathcal{P}_0^4}.$$

From here passing on to new distribution function $\mathcal{G}(\eta, x) \sim 1$:

$$f(\eta, x) = \frac{3\pi}{4(2S+1)}\frac{\mathcal{G}(\eta, x)}{\mathcal{P}_0^4}, \tag{76}$$

so that

$$\int\limits_0^\infty \mathcal{G}(\eta, x)x^2dx = 1, \tag{77}$$

and changing of variables, we obtain relations:

$$\beta(\eta) = \frac{b(\eta)}{\mathcal{P}_0}; \tag{78}$$

where

$$b(\eta) = \int\limits_0^\infty \mathcal{G}(\eta, x)xdx; \tag{79}$$

$$\int\limits_0^\infty \mathcal{G}(\eta, x)x^3dx = 1. \tag{80}$$

So, in selected variables normalized distribution function $\mathcal{G}(\eta, x)$ must satisfy the two normalization relations - (77) and (80).

Inserting renormalized time τ:

$$\tau = \frac{3A\eta}{8\pi\mathcal{P}_0^3}, \tag{81}$$

Finally we write diffusive equation (60) respect to function $\mathcal{G}(\tau, x)$ in the form of:

$$\frac{\partial\mathcal{G}}{\partial\tau} = \frac{1}{x^2}\frac{\partial}{\partial x}x^2\left(\frac{\partial\mathcal{G}}{\partial x} + 2b(\tau)\mathcal{G}\right). \tag{82}$$

Equation (82) must be solved with initial and boundary conditions in the form of:

$$\mathcal{G}(0, x) = G(x); \quad \lim_{x\to\infty} G(\tau, x)x^3 = 0, \tag{83}$$

in consequence of (75), (77) function $G(x)$ must satisfy the integral conditions:

$$\int\limits_0^\infty G(x)x^2dx = 1; \tag{84}$$

$$\int\limits_0^\infty G(x)x^3dx = 1. \tag{85}$$

The last of conditions (83) are necessary for providing of energy integral convergence. Setting

$$b(\tau) \approx \mathrm{Const},$$

what could be done at early stage of evolution, and separating variables in equation (82):

$$\mathcal{G}(\tau, x) = T(\tau)X(x),$$

we obtain following equation:

$$T = e^{-\lambda^2\tau}, \tag{86}$$

$$xX'' + 2X'(1+bx) + X(\lambda^2x + 4b) = 0. \tag{87}$$

If $X_\lambda(x)$ - solution of equation (87), then general solution of diffusive equation (82) can be written as:

$$\mathcal{G}(\tau, x) = \int\limits_0^\infty X_\lambda(x)e^{-\lambda^2\tau}d\lambda. \tag{88}$$

Solution of equation (87) is expressed through combination of hypergeometric functions $\Phi(\alpha, \gamma, z)$ and $\Psi(\alpha, \gamma, z)$ (or *Whittaker* functions), $M_{\mu,1/2}(z)$ and $W_{\mu,1/2}(z)$:

$$X_\lambda(x) = \frac{e^{-bx}}{x}\Big[C_1(\lambda)M_{-i\mu,1/2}(2i\sqrt{\lambda^2-b^2}x) +$$

$$+ C_2(\lambda)W_{-i\mu,1/2}(2i\sqrt{\lambda^2-b^2}x)\Big], \tag{89}$$

where

$$\mu = \frac{b}{\sqrt{\lambda^2-b^2}}.$$

Specifically, in case $b=0$ we get $\mu=0$,

$$X_\lambda(x)=\frac{1}{x}\left[C_1(\lambda)M_{0,1/2}(2i\lambda x)+C_2(\lambda)W_{0,1/2}(2i\lambda x)\right]$$

Using relations of Whittaker functions with modified Bessel functions $I_\mu(z)$ и $K_\mu(z)$ [22]:

$$W_{0,\mu}(z)=\sqrt{\frac{z}{\pi}}K_\mu\left(\frac{z}{2}\right);$$

$$M_{0,\mu}(z)=2^{2\mu}\Gamma(\mu+1)\sqrt{z}I_\mu\left(\frac{z}{2}\right), \tag{90}$$

and also relations of modified Bessel functions of imaginary argument with Bessel functions of the first and the third kind $J_\mu(z)$ и $H^{(2)}(z)$ (Hankel function) [23]:

$$I_\mu(z)=e^{\frac{i\pi}{2}}J_\mu\left(ze^{-\frac{i\pi}{2}}\right), \qquad (\forall\, z|-\frac{\pi}{2}<\arg(z)<\pi),$$

$$K_\mu(z)=-\frac{i\pi}{2}e^{-\frac{i\pi\mu}{2}}H^{(2)}\left(ze^{-\frac{i\pi}{2}}\right), \tag{91}$$

obtain from (89) solution in case of $\beta\to 0$:

$$X_\lambda(x)=\sqrt{2\lambda}x\left[C_1(\lambda)\sqrt{\pi}e^{i3\pi/4}J_{\frac{1}{2}}(\lambda x)+\right.$$

$$\left.+C_2(\lambda)e^{-i3\pi/4}\sqrt{1}\pi H^{(2)}_{\frac{1}{2}}(\lambda x)\right]. \tag{92}$$

Taking into account the fact, that Bessel functions with half-integer index are expressed through elementary functions:

$$J_{\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}\sin z; \quad H^{(2)}_{\frac{1}{2}}(z)=i\sqrt{\frac{2}{\pi z}}e^{-iz},$$

finally we get:

$$\mathcal{G}(\tau,x)=\int\limits_0^\infty P(\lambda)e^{-\lambda^2\tau+i\lambda x}d\lambda.$$

This result could be received directly from equation (82), if it is assumed that $b=0$, - then for function $\mathcal{G}(\tau,x)$ we get thermal conductivity equation:

$$\frac{\partial\mathcal{G}}{\partial\tau}=\frac{1}{x^2}\frac{\partial}{\partial x}x^2\left(\frac{\partial\mathcal{G}}{\partial x}\right), \tag{93}$$

which coincides with three-dimensional thermal conductivity equation in spherical coordinates in case of spherical symmetry. Standard solution of three-dimensional thermal conductivity equation, corresponding to initial condition:

$$U(x,y,z,0)=U_0(x,y,z) \tag{94}$$

is given by (see, e.g., [24])):

$$U(\vec{r},t)=\frac{1}{8\pi t^{3/2}}\int\limits_{V_0}U_0(\vec{r}_0)e^{-\frac{(\vec{r}-\vec{r}_0)^2}{4t}}dV_0 \tag{95}$$

Let's choose temporary coordinates in order to obtain spherically symmetric solution on basis of this solution:

$$\vec{r}_0=(0,0,r)$$

and in integral (95) we pass to spherical coordinates:

$$x=r_0\cos\phi\cos\theta;\, y=r_0\sin\phi\cos\theta;\, z=r_0\sin\theta$$

$$\Rightarrow(\vec{r}-\vec{r}_0)^2=r_0^2+r^2-2rr_0\sin\theta$$

and integrating in angular variables we get solution, that at early stages of expansion circumscribes normalized function of distribution in unitary limit:

$$\mathcal{G}(\tau,x)=\frac{1}{2x\sqrt{\pi\tau}}\times$$

$$\int\limits_0^\infty G(y)\left[e^{-\frac{(x-y)^2}{4\tau}}-e^{-\frac{(x+y)^2}{4\tau}}\right]ydy. \tag{96}$$

Note that in approximation $\beta\to 0$ function $\mathcal{G}$ must not satisfy integral relations of normalization (84), (85) by now in studied range, because this approximation doesn't consider processes of particles influx in examined area for an account of collisions with low-energy particles, that lead to thermalization of distribution. Nevertheless, initial distribution must satisfy these normalization relations.

Let's study evolution of high-energy tail of distribution. Assume that initial distribution was given by:

$$\mathcal{G}(0,x)=\frac{G_0k^3}{(k^2+x^2)^{3/2}}\chi(x_0-x),$$

where $\chi(x)$ - Heaviside function:

$$\chi(x)=\begin{cases}0, & x\leqslant 0,\\ 1, & x>0\end{cases}$$

i.e., at $x_0>x\gg k$ $\mathcal{G}(0,x)\approx G0/x^3$ - energy is distributed uniformly. Two constants k и G_0 must be determined from pair of normalization relations (84), (85). Hence, we obtain relations:

$$1=G_0k^3\left(\ln(\zeta+\sqrt{1+\zeta^2})-\frac{\zeta}{\sqrt{1+\zeta^2}}\right); \tag{97}$$

$$1=G_0k^4\left(\frac{\zeta^2+2}{\sqrt{1+\zeta^2}}-2\right), \tag{98}$$

where

$$\zeta=\frac{x_0}{k}.$$

Thus, for three constants G_0, x_0, k we have two equations (97), (98), here one parameter, for example, ζ, remains free. From here find:

$$G_0k^3=\frac{\sqrt{1+\zeta^2}}{\sqrt{1+\zeta^2}\ln(\zeta+\sqrt{1+\zeta^2})-\zeta}; \tag{99}$$

$$k = \frac{\sqrt{1+\zeta^2}\ln(\zeta+\sqrt{1+\zeta^2})-\zeta}{\zeta^2+2-2\sqrt{1+\zeta^2}}. \qquad (100)$$

Substitution of values (99), (100) in relation (96) brings to final formal equation, suitable for large values of momentum, - this solution is defined by free value ζ.

7. Numerical model of evolution of high-energy tail of distribution

Introducing values:

$$z = \frac{y}{k}; \qquad u = \frac{x}{k} \qquad (101)$$

and taking into account relation (99), (100) in relation (96), we get solution of in quadratures:

$$\mathcal{G}(u,\tau;\zeta) = \frac{1}{2u\sqrt{\pi\tau}}\frac{\sqrt{1+\zeta^2}(\zeta^2+2-2\sqrt{1+\zeta^2})^3}{[\sqrt{1+\zeta^2}\ln(\zeta+\sqrt{1+\zeta^2})-\zeta]^4}\times$$

$$\times\int\limits_0^\zeta \Big[e^{-\dfrac{k^2(z-u)^2}{4\tau}} - e^{-\dfrac{k^2(z+u)^2}{4\tau}}\Big]\frac{zdz,}{(1+z^2)^{3/2}} \qquad (102)$$

where it is necessary to substitute value $k = k(\xi)$ from (100).

Let's analyze received solution by numerical methods using comprehensive computer system Maple.

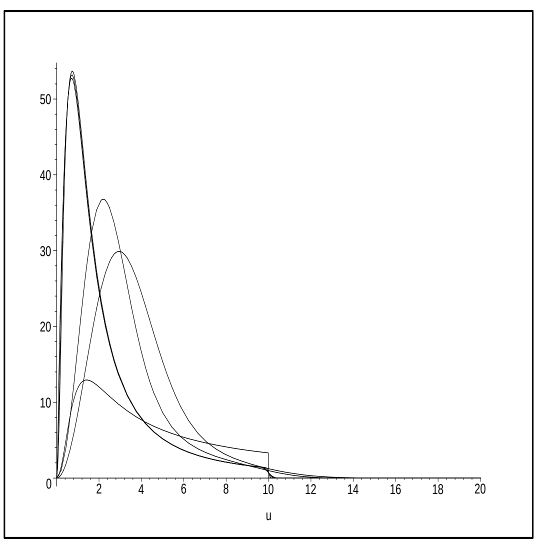

Рис. 1. *Evolution of numerical density of superthermal particles $x^2\mathcal{G}(u,\tau,10)$ at little times for values τ =0; 0,0001; 0,0005; 0,001; 0,05; 0,01. On abscissa scale u.*

We will get estimate for asymptotic behavior of distribution tail in the range of very high energies

$$u \gg 1. \qquad (103)$$

In range (103) integrand function in (102) is given by narrow peak with vertex point in range $z = u$, and at the same time the second exponential member in integrand expression vanishingly small. If in this case $u > \zeta$, integral becomes exponentially small. Even though u, ζ, integral can be estimated as area of peak, i.e., as product of it's altitude and half-width. The maximum of integrand function is:

$$\max\Phi(z=u) = \frac{u}{(u^2+1)^{3/2}} \simeq \frac{1}{u^2}.$$

For calculation of half-width $2\Delta z$ of peak we have following equation:

$$\frac{1}{2u^3} = e^{-\frac{k^2\Delta z}{4\tau}}\frac{1}{u^3},$$

wherefrom we find:

$$2\Delta z = \frac{4\sqrt{\tau\ln 2}}{k}.$$

So, we get estimate:

$$\mathcal{G} \sim \begin{cases} \frac{1}{x^3} & x < x_0; \\ e^{-x^2/4\tau}, & x > x_0. \end{cases} \qquad (104)$$

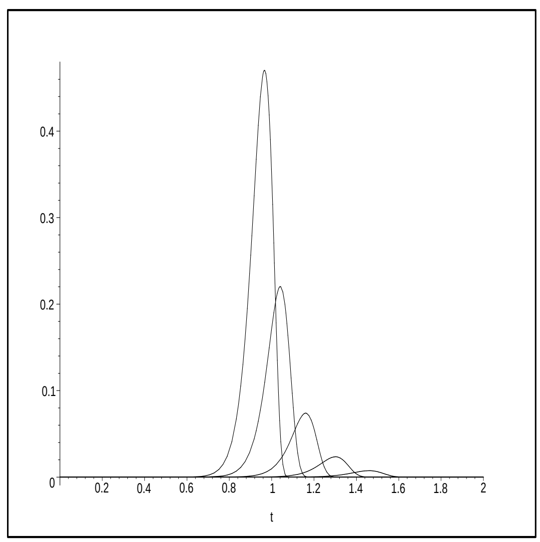

Рис. 2. *Evolution of numerical density of superthermal particles in high-energy tail $u^2\mathcal{G}(u,\tau,1)$ for values τ =0,1; 1; 10; 100; 1000 (from left to right). On abscissa scale $\log_{10} u$.*

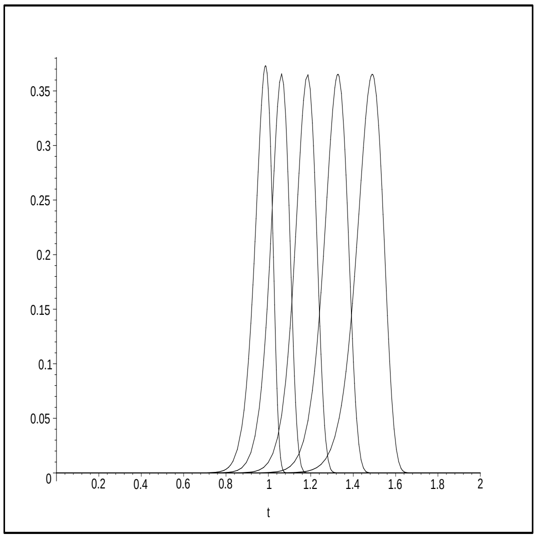

Рис. 2. *Evolution of energy distribution of superthermal particles in high-energy tail $u^2\mathcal{G}(u,\tau,1)$ for values τ =0,1; 1; 10; 100; 1000 (from left to right). On abscissa scale $\log_{10} u$.*

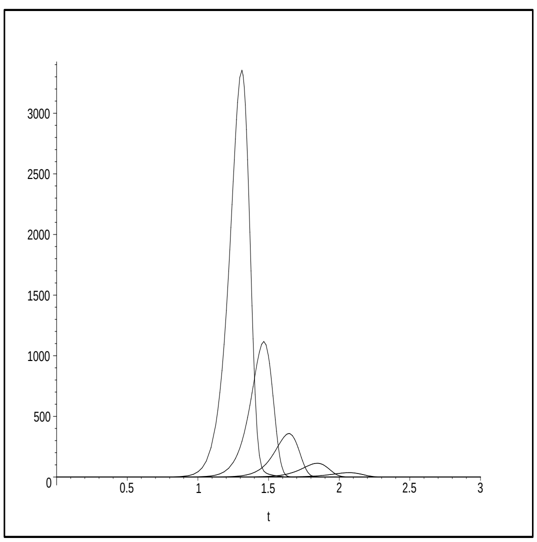

Рис. 2. *Evolution of numerical density of superthermal particles in high-energy tail $u^2\mathcal{G}(u,\tau,100)$ for values τ =0,1; 1; 10; 100; 1000 (from left to right). On abscissa scale $\log_{10} u$.*

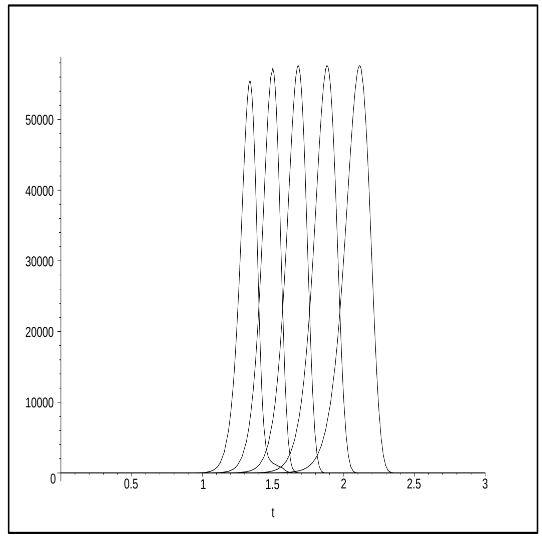

Рис. 2. *Evolution of energy distribution of superthermal particles in high-energy tail $u^2\mathcal{G}(u,\tau,100)$ for values τ =0,1; 1; 10; 100; 1000 (from left to right). On abscissa scale $\log_{10} u$.*

8. Conclusions

From carried results we see that conformal energy of distribution tail increases with time. This effect has analogy to well-known effect of escaping particles. Total energy of particles doesn't preserve, i.e. our model doesn't consider particles interactions in low energy ranges. In more total model at the same time with increasing of particles energy in tail must decrease energy of distribution in ranges of average energies. In next article we will consider more complete model of evolution of superthermal particles, based on integro-differential equation.

Список литературы

[1] Yu.G.Ignat'ev. Izvestiya Vuzov, Fizika, 1979, V. 22, No.2, p. 72 (in Russian)

[2] Yu.G.Ignat'ev. "Problems of gravitation theory and elementary particles", Atomizdat, Moscow, 1980, No. 11, p. 113 (in Russian)

[3] Yu.G.Ignat'ev. Izvestiya Vuzov, Fizika, 1980, V. 23, No.8, p. 42 (in Russian)

[4] Yu.G.Ignat'ev. Izvestiya Vuzov, Fizika, 1983, V. 24, No.8, p. 19 (in Russian)

[5] Yu.G.Ignat'ev. Izvestiya Vuzov, Fizika, 1982, V. 25, No. 3, p. 92 (in Russian)

[6] H.M.Pilkuhn. *Relativistic Particle Physics*, Springer-Verlag, New York Inc, 1979

[7] Yu.G.Ignat'ev. *Kinetic methods in relativistic theory of gravitation*, Thesis for a Doctor's degree, Kazan, 1986 (in Russian)

[8] Okun L.B. *Leptons and quarks*, Moscow, Nauka, 1981 (in Russian)

[9] Froissart M. *Phys. Rev.*, 1961, V. 123, p.1053

[10] Martin A. *Phys. Rev.*, 1963, V. 129, p. 1432

[11] Martin A. *Nuovo. Cim.*, 1966, V. 142, p. 930

[12] Jin Y.S., Martin A. *Phys. Rev.*, 1964, V. 135B, p. 1369

[13] Sugawara M. *Phys. Rev. Lett.*, 1965, V. 14, p. 336

[14] R.I.Eden, *High Energy Collisions of Elementary Particles*, Cambridge At the University Press, 1967

[15] Bjorken J.D., Paschos E.A. *Phys. Rev.*, 1969, V.185, p. 1975

[16] Feyman R.P. *Phys. Rev. Lett.*, 1969, V.23, p. 1415

[17] Cabibo N., Parivisi G., Tesla M. *Lett. Nuovo. Cimento*, 1970, V. 4, p.35

[18] L.P.Gritchuk, *JETF*, 1974, e.67, p. 825 (in Russian)

[19] Yu.G.Ignat'ev. Reports of VI-th Soviet conference of gravitation, Moscow, 1984, p. 18 (in Russian)

[20] Yu.G.Ignat'ev. *Izvestiya Vuzov, Fizika*, 1986, V.29, No. 2, p. 27 (in Russian)

[21] Yu.G.Ignat'ev. *Problems of gravitation theory, relativistic kinetics and evolution of the Universe*, Izdatelstvo KGPI, Kazan, 1988, p. 62 (in Russian)

[22] I.S.Gradstein, I.M.Ryzhik. *Table of integrals, sums, sequenses and products*, Moscow, Fizmatgiz, 1963 (in Russian)

[23] N.N.Lebedev. *Special functions and its applications*, Moscow, GIFML, 1963 (in Russian)

[24] S.L.Sobolev. Equatuons of mathematical physics, Moscow, Nauka, 1966 (in Russian)