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Quantum variation measurement of a force

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Abstract

The quantum limit for the resolution of a small force using an optical transducer of a displacement with a coherent nonmodulated pump is proved to be less than the standard quantum limit if one measures the quadrature amplitude in the output wave, squeezed by the ponderomotive nonlinearity mechanism. This squeezing has a spectral dependence and we propose a procedure showing it.

1. Introduction

Quantum noise in a mechanical displacement sensor is the key problem for the interferometric gravitational-wave antenna (the LIGO project) and in some other fundamental experiments. Under a continuous coordinate measurement the back-action noise of the sensor is responsible for the limit of the sensitivity [1–3], known as the standard quantum limit (SQL). For the simplest optical transducer (Fig. 1) the back-action noise is caused by fluctuations of the ponderomotive force of light pressure: the amplitude fluctuations of the incident wave transform to phase fluctuations of the reflected one, and the conversion factor is proportional to the square root of the pump power (the power of the incident wave).

The SQL for a coordinate is usually used to determine the value of the minimum registered force. For the force $F_S = F\sin(\omega_F t)$, acting on the free mass m during a time t_F, which is a multiple of $2\pi/\omega_F$, the SQL is equal to $F_{SQL} \simeq \sqrt{m\hbar\omega_F^2/t_F}$. Here and below we assume that there are no intrinsic mechanical noise and losses.

A force measurement with an error less than F_{SQL} is known to be theoretically possible in a continuous coordinate measurement if the noise of the meter is correlated in a special way [2,3]. One of the possible ways to break through the SQL is the use of a modulated pump [4], frequency anticorrelated states [5] or a pump in a squeezed state [6] (this squeezing should have a special spectral dependence; it is not clear how this can be realized).

It was shown [7–9] that the SQL can be overcome even with a coherent nonmodulated pump – without using squeezed states, photon number states or any nonclassical states. For this one has to measure not the phase but a specially chosen *squeezed* quadrature amplitude B of the reflected wave (see Fig. 2). It is the light pressure mechanism (in fact – a ponderomotive kind of $\chi^{(3)}$ susceptibility) causing a back action that leads to *squeezing* of reflected light. Registration of $B(\vartheta)$ allows one "not to see" the back-action perturbation.

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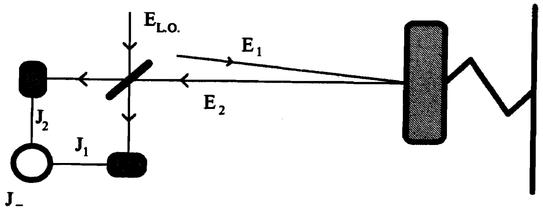

Fig. 1. Simplest optical transducer of a mechanical displacement as a detector of the variation of the coordinate of the oscillator caused by the signal force. The plane em wave E_1 is normal to the surface of an ideal mirror without absorption. The action of a signal force changes the coordinate of the mirror. It causes a phase shift of the reflected wave E_2, which is detected in a modified balanced homodyne scheme. The differential photocurrent $J_- = J_1 - J_2$ is proportional to the quadrature amplitude $B(\vartheta(t))$ of the signal wave and the phase $\vartheta(t)$ is determined by the phase of the local oscillator E_{LO}. To eliminate the fluctuations of the quadrature amplitudes of the local oscillator wave one should use large $E_{LO}(t)$: $|E_{LO}| \gg K|E_2|$ (here $|E_{LO}|, |E_2|$ are the amplitudes of the electrical fields, $K > 1$ is the coefficient of the squeezing of the signal field E_2), also we assume the local oscillator field to be in a coherent state.

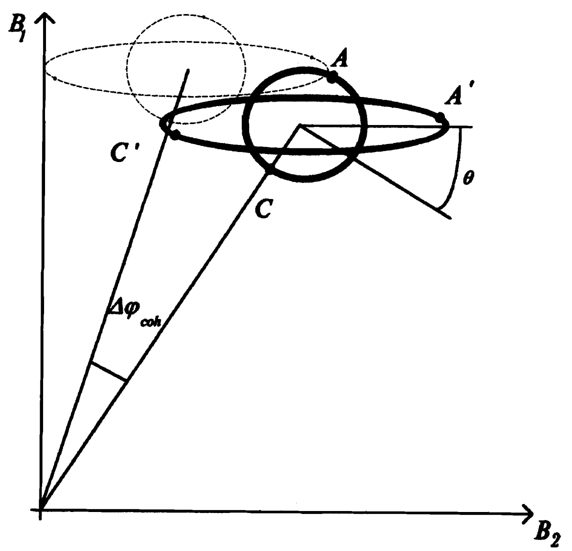

Fig. 2. If the incident wave is in a coherent state, on the phase diagram its fluctuations are described by a round spot, rotating at a distance $\sqrt{n}$ from the center of coordinates: the uncertainties of the quadrature amplitudes $\langle \Delta B_1^2 \rangle$ and $\langle \Delta B_2^2 \rangle$ are equal. The ponderomotive nonlinearity (light pressure mechanism) produces a phase–amplitude correlation in the reflected wave. This means the squeezing of the Fourier transform of the quadrature amplitude $B(\Omega, \vartheta)$ – the circle turns into an ellipse. The points A and C for the incident wave transform to the points A′ and C′, respectively, for the reflected wave. The amplitude–phase correlation appears – the reflected wave is in the squeezed state. The action of the minimal signal force F_{min} to be registered has to change the phase of the reflected wave by the value of the phase uncertainty of the incident wave $\Delta\varphi_{coh}$, but only if one measures the quadrature amplitude $B(\vartheta)$, which coincides with B_1 in this figure. Then the force F_{min} may be less than F_{SQL}, because $\Delta\varphi_{coh} \simeq 1/2\sqrt{n}$ and the error of the force measurement decreases with increasing pump power. The two states of the reflected wave with and without action of the signal force are shown by solid and dashed lines.

In our case the squeezing of the reflected wave has a special feature: ϑ depends on the spectral frequency $\vartheta = \vartheta(\Omega)$. Therefore the usual homodyne scheme (Fig. 1) (with a constant phase of the local oscillator) is suitable for measureming the squeezed quadrature amplitude only in a narrow frequency band inside which one can neglect the dependence $\vartheta = \vartheta(\Omega)$. In Section 2 we propose modulating the phase of the local oscillator during the averaging time in order to measure the spectral-dependent squeezing in a wide frequency band (i.e. a short averaging time, which is required in the LIGO experiment). We show that by choosing the phase modulation and the averaging function in a proper way the back-action noise can be fully compensated. Thus one can conclude that the error of the force measurement decreases with an increase of the pump power.

However, there is a physical mechanism that limits the sensitivity in the force measurement with increasing pump power – radiative friction. (If the mirror moves in the same direction as the incident wave, the flux of incident photons becomes smaller because the light path lengthens. Thus the light pressure force depends on the velocity of the mirror.) In Section 3 we show that this effect puts the limit for squeezing and therefore defines the minimal error of the force measurement: $F_{min} \simeq \xi F_{SQL}\sqrt{\omega_F/\omega_o}$ (ω_o is the frequency of the optical pump, ξ the numerical factor of order unity), which can be achieved at a certain optimal pump power. This result however is better than the minimal error obtained in Refs. [8,9]: $F_{min.} \simeq \xi F_{SQL}(\omega_F/\omega_o)^{1/4}$, where the possibility of full compensation of the back action was overlooked.

The proposed procedure is not a quantum nondemolition (QND) measurement – there is no nondisturbed variable of the mechanical oscillator. Instead, the reflected wave brings out little information about the coordinate, momentum or their combination, because they are strongly disturbed by the measuring device. Only *variation of the coordinate* caused by the signal force is registered [10]. Such procedures may be called *quantum variation measurements*. We can conclude that the problem of detecting a signal action and the problem of the QND measurement are different. Each of them has its own strategy. In general one is not related to the other.

2. Measurement of the quadrature amplitude

The differential photocurrent J_- in the homodyne scheme (Fig. 1) is proportional to the quadrature component $B(\vartheta(t),t)$ of the reflected wave: $J_-(t) = J_0 B(\vartheta(t),t)$, with the angle $\vartheta(t)$ determined by the phase of E_{LO} (here J_0 is constant). However, the measurement is not instantaneous and includes averaging. So an experimentalist measures the value

$$B_T = \int_0^T \Phi(t) B(\vartheta(t),t)\,dt \tag{1}$$

and in doing so has the opportunity to choose the averaging function $\Phi(t)$ and the modulation of the local oscillator phase $\vartheta(t)$. Analyzing the noise of the value B_T one can conclude that (see the Appendix) the back action can be fully compensated under the following condition (we write it in both time and spectral domains),

$$g_C(t) + \int_t^T g_S(\tau) K(\tau - t)\,d\tau = 0, \tag{2}$$

$$g_C(-\Omega) + K(\Omega) g_S(-\Omega) = 0, \tag{3}$$

where

$$g_S(t) = \Phi(t)\sin\vartheta(t), \quad g_C(t) = \Phi(t)\cos\vartheta(t)$$

and

$$g_S(\Omega) = \int_0^T g_S(t)\,e^{i\Omega t}\,dt,$$

$$g_C(\Omega) = \int_0^T g_C(t)\,e^{i\Omega t}\,dt$$

are their Fourier transforms,

$$K(\Omega) = 4\omega_o\delta_R/(\omega_M^2 - \Omega^2 - 2i\delta_R\Omega),$$

where $\delta_R = 2W/mc^2$ is the coefficient of the radiative friction of the mechanical oscillator, W is the mean power of the incident wave, $K(t) = (2\pi)^{-1}\int_{-\infty}^{\infty} K(\Omega)\,e^{-i\Omega t}\,d\Omega$.

Under conditions (2), (3) B_T has the following simple form,

$$- B_T = \int_0^T g_S(t)\Big(A_{ph}(t) + \int_{-\infty}^{t} d\tau K(t-\tau) F_S(\tau)/\sqrt{4\pi m\delta_R\hbar\omega_o}\Big)\,dt$$

$$= (2\pi)^{-1}\int_{-\infty}^{\infty} g_S(-\Omega)\left[(a - a_-^+)/i + K(\Omega) F_S(\Omega)/\sqrt{4\pi m\delta_R\hbar\omega_o}\right] d\Omega. \tag{4}$$

Here $a(\omega), a^+(\omega)$ are the annihilation and creation operators, describing quantum fluctuations of the incident wave (their commutators are $[a(\omega)a^+(\omega')] = \delta(\omega - \omega')$, the averages $\langle a^+(\omega)a(\omega')\rangle = 0$, and we denote $a = a(\omega_o + \Omega), a_-^+ = a^+(\omega_o - \Omega)$), A_{ph} are the phase fluctuations of the incident wave (we assume the mean amplitude of the incident wave to be real),

$$A_{ph}(t) = -\frac{i}{2\pi}\int_{-\infty}^{\infty} (a - a_-^+)\,e^{-i\Omega t}\,d\Omega,$$

$F_S(t)$ is the signal force, $F_S(\Omega) = \int_{-\infty}^{\infty} F_S(t)\,e^{i\Omega t}\,dt$ is its Fourier transform.

The sensitivity of the proposed scheme is determined by the achievable value of the squeezing $\langle\Delta B_T^2\rangle$. For the measurement at frequency Ω_o (in the narrow frequency bandwidth) one can obtain from (4):

$$\sqrt{\langle\Delta B_T^2\rangle} \simeq \sqrt{\langle\Delta B_{T\mathrm{coh}}^2\rangle}/|K(\Omega_o)|,$$

where the dispersion $\langle\Delta B_{T\mathrm{coh}}^2\rangle$ is calculated for coherent light. The squeezing is restricted because the coefficient $|K(\Omega_o)|$ is limited by the maximum value $2\omega_o/\Omega_o$, which is achieved at $W \to \infty$. This limit is the result of radiation friction introduced by the measuring device.

3. Detection of the force

To evaluate the minimal detectable force with an a priori known waveform (template), let us apply the recipes of the theory of optimal filtration to the sufficient statistic B_T (4). The signal-to-noise ratio is

$$\mu = \int_{-\infty}^{\infty} \frac{|F_S(\Omega)|^2 4\omega_o \delta_R \, d\Omega}{\pi m\hbar[(\omega_M^2 - \Omega^2)^2 + 4\delta_R^2 \Omega^2]}.$$

Within the free mass approximation ($\Omega_F \gg \omega_M$) this integral has a maximum at the optimal pump,

$$\delta_{R\mathrm{opt}} = I_1 \omega_F,$$

and the minimal detectable amplitude of force is equal to

$$F_{\min} = I_2 F_{\mathrm{SQL}} \sqrt{\omega_F/\omega_o}. \quad (5)$$

Here I_1, I_2 are constants of order unity. We should note that the maximum sensitivity is predicted for large pump values that may not be realized in the experiment. (The condition $\delta_R \simeq \omega_F$ means that the signal force and the force of radiative friction are of the same order.) In the scheme using a Fabry–Perot resonator with a resonance pump the requirement on the pump is somewhat easier, $W_{\mathrm{opt}} \simeq \frac{1}{32} mc^2 \omega_F [R^2 + (4L\omega_F/c)^2]$, but also is hardly realizable. Here R is the transmissivity of the input mirror (losses are absent in both mirrors), L is the separation between mirrors. (This formula is obtained under the assumption that the rigidity, introduced by the optical resonator into the mechanical system, is compensated.) Note that $W_{\mathrm{opt}} \simeq W_{\mathrm{SQL}}(\omega_o/\omega_F)$, where W_{SQL} is the value of the pump power at which the SQL is achieved (the same value is projected in LIGO). For $m = 1$ kg, $\omega_F = 10^2$ s^{-1}, $L = 4$ km, $\omega_o = 10^{15}$ s^{-1}, $R = 10^{-5}$, the value of W_{SQL} is about 1 W and $W_{\min} \simeq 10^{13}$ W! One can conclude that with a reasonable pump power $W_{\mathrm{SQL}} \ll W \ll W_{\mathrm{opt}}$, the error of the force measurement monotonically drops with increasing pump power: $F \simeq F_{\min}\sqrt{W_{\mathrm{opt}}/W} \simeq F_{\mathrm{SQL}}\sqrt{W_{\mathrm{SQL}}/W}$. For example, with $W = 10^4$ W one can obtain the estimation $F \simeq 10^{-2} F_{\mathrm{SQL}}$.

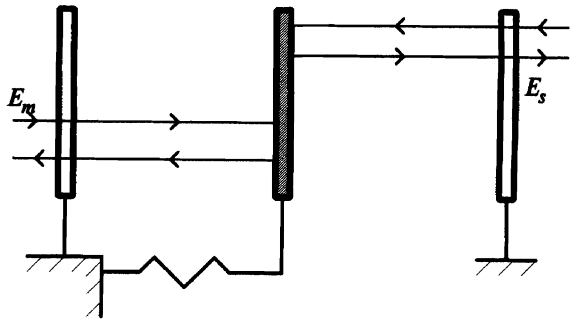

Fig. 3. The scheme of the QND measurement of the number of quanta using the principle of quantum variation measurement. The meter wave E_m serves to registrate the displacement of the movable mirror caused by the ponderomotive light pressure force of the signal wave E_S.

There is also another cause why the large value of the pump power is not worth using. All the above considerations are based on a *linear* expansion in terms of $\sim \omega_o X/c$ and $\sim a/\sqrt{W}$. It is only in this approximation that the distribution of fluctuations in the reflected wave is described by the regular ellipse shown in Fig. 2. However taking account of the quadratic terms $\sim (\omega_o X/c)^2$ and $\sim (a/\sqrt{W})^2$ results in the bending of the ellipse and consequently in the increase of the uncertainty of the squeezed quadratic amplitude. One can neglect this deformation under the condition: $|K|^3 \sqrt{\hbar\omega_o\omega_F/64W} \ll 1$, which is not valid for W_{opt} and the parameters listed above. It is worth noting that this restriction is not a principal one but is connected with the linear procedure of the measurement of the quadratic amplitude, which allows one "to distinguish regular ellipses". To overcome this obstacle one has to modify the linear procedure into a nonlinear one in such a way so as "to distinguish the bent ellipses". The discussion of such a procedure will be the subject of a later publication.

The above described measurement procedure can be used for a QND measurement of the number of optical quanta, if one has another resonator on the opposite side of a movable mirror (Fig. 3). The wave in the right-hand side resonator (let its frequency, number of quanta and duration be ω_o, n, τ, respectively) produces a pressure force that is registered by a meter wave in the left resonator. From (5) one can obtain a measurement error Δn of the quantum number (we assume that $L\Omega/c \ll R$),

$$\Delta n = \frac{R}{\omega_0 \tau} \sqrt{\frac{mc^2}{\hbar\omega_0}}.$$

With $\omega_0 = 10^{15}\ \mathrm{s}^{-1}$, $\tau = 0.01$ s, $R = 10^{-5}$, $m = 0.1$ g we obtain the numerical estimation $\Delta n \simeq 10^{-2}$. Therefore one can detect a single quantum without its demolition. Note that the main obstacles for the experimental realization of this scheme are losses in the resonator mirrors and the intrinsic thermal noise of the mechanical oscillator.

4. Conclusion

The main advantage of the proposed measurement scheme in the authors' view is that it allows one to overcome the SQL in the usual scheme of the continuous measurement of a coordinate with a usual coherent laser pump (as planned in the LIGO) experiment only by adding the modified homodyne scheme with a phase modulated local oscillator. However, it can be shown that for a squeezed pump with small phase uncertainty $\Delta\varphi \ll \Delta\varphi_{\mathrm{coh}}$ the error ΔF of the force measurement may be considerably less than (5): $\Delta F \simeq F_{\min}\Delta\varphi/\Delta\varphi_{\mathrm{coh}}$ or the error $\Delta F = F_{\min}$ can be obtained with a smaller pump power. However, the squeezed pump is hard to realize in an experiment.

It should be mentioned that the above considerations are valid for mirrors with zero losses. In the presence of losses the sensitivity for the transducer of the displacement with an optical Fabry–Perot resonator (or with a Michelson interferometer, as planned in the LIGO) in the force measurement is limited by the value $F_{\mathrm{loss}} \simeq \zeta F_{\mathrm{SQL}}(R/Q)^{1/4}$ [9], where R is the sum of the loss coefficients in all mirrors and Q is the transmission coefficient of the input mirror, ζ is about unity.

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Appendix A

The coordinate of the mechanical oscillator $X(t)$ is changed by the action of the signal force $F_S(t)$ and the fluctuating part of the light pressure (back action) as follows (see the notations in Section 3) [7,8],

$$X(\Omega) = (a + a_-^+)\frac{K(\Omega)}{2}\sqrt{\frac{\pi\hbar c^2}{2W\omega_0}} + \frac{F_S(\Omega)}{m(\omega_M^2 - \Omega^2 - 2i\delta_R\Omega)}, \tag{A.1}$$

where $X(\Omega) = \int_{-\infty}^{\infty} X(t)\,e^{i\Omega t}\,dt$ is the Fourier transform of $X(t)$. The annihilation operators describing the quantum fluctuations of the reflected wave $b(\omega)$ (commutators, averages and all notations are as for $a(\omega)$) are

$$b = -a - iX(\Omega)\sqrt{\frac{2W\omega_0}{\pi\hbar c^2}}. \tag{A.2}$$

The quadrature amplitude $B(\vartheta(t), t)$ of the field E_2 in the reflected wave is then expressed as follows,

$$B(\vartheta(t), t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\Omega t}[b\,e^{-i\vartheta(t)} + b_-^+\,e^{i\vartheta(t)}]\,d\Omega.$$

Using (A.1), (A.2), B_T can be written in the time domain in terms of the functions $g_C(t)$ and $g_S(t)$,

$$-B_T = \int_0^T \left[A_{\mathrm{am}}(t)g_C(t) + g_S(t)\int_{-\infty}^{t} d\tau\,K(t-\tau)A_{\mathrm{am}}(\tau)\right]dt + \int_0^T g_S(t)\left(A_{\mathrm{ph}}(t) + \int_{-\infty}^{t} d\tau\,\frac{K(t-\tau)F_S(\tau)}{\sqrt{4\pi m\delta_R\hbar\omega_0}}\right)dt, \tag{A.3}$$

where the mean amplitude of the incident wave is assumed to be real and amplitude fluctuations are introduced,

$$A_{\rm am}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (a + a_-^+)\,{\rm e}^{-{\rm i}\Omega t}\,{\rm d}\Omega.$$

B_T may also be written in the spectral domain,

$$\begin{aligned} -B_T = {} & \frac{1}{2\pi}\int_{-\infty}^{\infty} (a + a_-^+) \\ & \times [g_{\rm C}(-\Omega) + K(\Omega) g_{\rm S}(-\Omega)]\,{\rm d}\Omega \\ & + \frac{1}{2\pi}\int_{-\infty}^{\infty} g_{\rm S}(-\Omega)\Big((a - a_-^+)/{\rm i} \\ & + \frac{K(\Omega)F_{\rm S}(\Omega)}{\sqrt{4\pi m\delta_{\rm R}\hbar\omega_{\rm o}}}\Big)\,{\rm d}\Omega. \end{aligned} \tag{A.4}$$

It is easy to show that when under conditions (2), (3) the noise of B_T is reduced to a minimum the corresponding expressions in square brackets in (A.3), (A.4) vanish. The noise of B_T depends only on the phase noise of the incident wave (4). This means that the back action is fully compensated during the averaging process through a proper choice of $\Phi(t)$ and $\varphi(t)$.

An important point is that $g_{\rm C}(-\Omega)$ and $g_{\rm S}(-\Omega)$ may be complex functions. If one restricts oneself to real functions (as in Refs. [7,8,11]), compensation cannot be full because the imaginary part of K results from radiative friction. It is this incomplete compensation that leads to the result $F_{\rm min} \simeq \xi F_{\rm SQL}(\omega_{\rm F}/\omega_{\rm o})^{1/4}$ instead of $F_{\rm min} \simeq \xi F_{\rm SQL}\sqrt{\omega_{\rm F}/\omega_{\rm o}}$.

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