

LINEAR-FRACTIONAL PROGRAMMING THEORY, METHODS, APPLICATIONS AND SOFTWARE

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*This book is dedicated to the
memory of my parents
Mihri Makmutova and
Bakish Bajalinov*

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Preface

This is a book on *Linear-Fractional Programming* (here and in what follows we will refer to it as "LFP"). The field of LFP, largely developed by Hungarian mathematician B. Martos and his associates in the 1960's, is concerned with problems of optimization.

LFP problems deal with determining the best possible allocation of available resources to meet certain specifications. In particular, they may deal with situations where a number of resources, such as people, materials, machines, and land, are available and are to be combined to yield several products. In linear-fractional programming, the goal is to determine a permissible allocation of resources that will maximize or minimize some specific showing, such as profit gained per unit of cost, or cost of unit of product produced, etc.

Strictly speaking, linear-fractional programming is a special case of the broader field of *Mathematical Programming*. LFP deals with that class of mathematical programming problems in which the relations among the variables are linear: the constraint relations (i.e. the *restrictions*) must be in linear form and the function to be optimized (i.e. the *objective function*) must be a ratio of two linear functions.

At the same time LFP includes as a special case the well known and widespread *Linear Programming* (LP). In the problems of LP both the restrictions and the objective function must be linear in form. If in an LFP problem the denominator of the objective function is constant, which equals to 1, then we have an LP problem. Conversely, any problem of LP may be considered as an LFP one with the constant denominator of the objective function.

In a typical maximum problem, a manufacturer may wish to use available resources to produce several products. The manufacturer, knowing how much profit and cost are made for each unit of product produced, would wish to produce that particular combination of products that would maximize the profit gained per unit of cost.

The example of a minimum problem is as follows: A company owning several mines with varying grades of ore is given an order to supply certain quantities of each grade; how can the company satisfy the requirements in such a way that the cost of unit of the ore is minimized?

Transportation problems comprise a special class of linear-fractional programming. In a typical problem of this type the trucking company may be interested in finding the least expensive (minimum total cost in LP or minimum cost per unit of transported product in LFP) way of transporting each unit of large quantities of a product from a number of warehouses to a number of stores.

Assignment problems are related to transportation problems. A typical example of this type of problem is finding the best way to assign n applications to n jobs, given ratings of the applicants with respect to the different jobs.

This book will deal with the study of the types of problems described above. The emphasis will be on formulating the problem, mathematically analyzing and finally solving it, and then interpreting the solution. Some special advanced topics of LFP will be considered too.

The main computational technique in linear-fractional programming is the simplex method developed by George B. Dantzig in the 1940's for solving linear programming problems and later, in 1960 upgraded by Béla Martos for solving LFP problems.

This book is completely self-contained, with all the necessary mathematical background given in Chapter 2. Readers who are familiar with linear algebra may omit this chapter. Knowledge of LP is desirable but not necessary.

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